ON RINGS OF QUOTIENTS

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Dedicated
To my
Beloved Parents

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Certificate

This is to certify that the dissertation entitled ***"On Rings of Quotients"*** has been written by ***Ms. Ambreen Bano*** under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh as a partial fulfillment for the award of ***Master of Philosophy in Mathematics.*** To the best of my knowledge, the exposition has not been submitted to any other university/institution for the award of the degree.

It is further certify that ***Ms. Ambreen Bano*** has fulfilled the prescribed conditions of duration and nature given in the ordinance of the Aligarh Muslim University, Aligarh.

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Preface

The theory of rings of quotients has its origin in the work of O. Ore and K. Asano in 1950's and 40's. But the subject did not really develope untill the end of 1950's when a number of important papers appeared by R.E.Johnson, Y.Utumi, A.W.Goldie, P.Gabriel and others. The most closest example of a ring of quotients is the field of fractions Q of a commutative integral domain R. It may be characterized by the two properties:
(i) For every $q \in Q$ there exists a nonzero $r \in R$ such that $qr \in R$.
(ii)Q is the maximal over ring of R satisfying condition (i).

The well-known construction of Q can be immediately extended to the case when R is an arbitrary commutative ring and S is a multiplicatively closed set of non zero divisors of R. In that case one defines the ring of fractions $Q = RS^{-1}$ as the ring consisting of pairs (r, s) with $r \in R$ and $s \in S$, with $(r, s) = (r', s')$ if $s'r = sr'$. The resulting ring Q satisfies (i), with the extra requirement that $r \in S$ and (ii). If R is a noncommutative domain, a related issue is that of embedding R into a division ring. In contrast to the commutative situation such embeddings need not always exist. If S is a multiplicatively closed set of nonzero divisors of R, then a right ring of fractions RS^{-1} can be defined in the same way as in the commutative case but for this to work one has to assume that S satisfies the following condition.
$(*)$ For each $r \in R$ and $s \in S$, there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$.

Every element in S becomes invertible in RS^{-1} and the elements of RS^{-1} may be written as rs^{-1} with $r \in R$ and $s \in T$. In particular, when T consists of all the nonzero divisors of R, the condition $(*)$ is called the Ore condition and RS^{-1} is called the right classical ring of quotients.

The present dissertation entitled ***"On rings of quotients"*** has been completed under the able guidance of ***Dr.Asma Ali***, comprises of four chapters. *Chapter I* contains some preliminary notions, basic definitions and important well-known results needed for the development of the subsequent texts.

Chapter II of the dissertation deals with the general introduction to the theory of rings of quotients (rings of fractions) in the setting of noncommutative rings.

Chapter III is devoted to the study of maximal rings of quotients. In 1956 Utumi [67] defined the maximal ring of quotients of a ring as follows: An over ring

Q of a ring R is said to be a right ring of quotients of R if R_R is a dense submodule of Q_R. The maximal right ring of quotients of R denoted by $Q^r_{max}(R)$ is the largest right ring of quotients of R. Analogously maximal left ring of quotients of R can be defined. There exist examples of ring R with over rings Q such that Q strictly contains the classical ring of quotients of R, but still Q may be viewed as a kind of general ring of quotients of R. This leads us to study of Findlay, Utumi, Lambek theory of maximal rings of quotients. Utumi [67] showed that the maximal right ring of quotients always exist. In case R is a commutative domain with quotient field K, we have of course $Q^r_{cl}(R) = Q^r_{max}(R) = K$. Another point of view about maximal rings of quotients is given by Lambek [48]. He related the maximal ring of quotients theory with injective module and pointed out that the maximal ring of quotients could be interpreted as the bicommutator of the injective envelope of R. After discussing maximal rings of quotients, it is natural to include an introduction to the idea of Martindale ring of quotients which was introduced by Martindale [51] for two sided ideals of a prime ring.

Chapter IV is devoted to the study of generalized polynomial identities (GPI) with coefficients in Utumi ring of quotients and Martindale ring of quotients. It is a generalization of a polynomial identity (PI) in which the coefficients come from the base field. A generalized polynomial identity of an algebra A over a field F is a polynomial expression f in noncommuting indeterminates and fixed coefficients from A between the indeterminates such that f vanishes upon all substitutions by elements of A. The theory of GPI was initiated by Amitsur [2] in 1965. Later on Martindale [51] extended Amitsur's work to prime GPI rings.

CHAPTER I

Preliminaries

1.1 Introduction

The present chapter is devoted to review some basic notions, important terminologies and known results in ring and module theory which we shall need for the developement of the subject in the subsequent chapters of the present dissertation. Suitable examples and necessary remarks are given at the proper places. The material for the present chapter has been collected mostly from the standard books like Lam [43], Beidar, Martindale and Mikhalev [5], Lambek [47], Jacobson [30], Herstein [26, 27], Rowen [63], McCoy [53], Goodearl [20], Passman [55].

1.2 Some ring and module theoretic notions

Definition 1.2.1 (Ideal) An additive subgroup I of a ring R is said to be a left (resp. right) ideal of R, if $ra \in I$ (resp. $ar \in I$) for all $a \in I$, $r \in R$. I is said to be an ideal of R if it is a left as well as a right ideal of R.

Example 1.2.1 Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$.
Then $I_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ is a right ideal but not a left ideal of R and $I_2 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ is a left ideal but not a right ideal of R.

Definition 1.2.2 (Regular element) An element a in a ring R is said to be regular if it is neither a left nor a right zero divisor of R.

Definition 1.2.3 (Commutator ideal) The commutator ideal $C(R)$ of a ring R is the ideal generated by all commutators $[x, y]$ with x, y in R.

Definition 1.2.4 (Nilpotent element) An element a of a ring R is said to be nilpotent if there exists a positive integer n such that $a^n = 0$, where a^n stands for $\underbrace{a.....a}_{n\text{-times}}$.

Definition 1.2.5 (Nilpotent ideal) A right (left, two sided) ideal A of a ring R is said to be a nilpotent ideal if there exists a positive integer $n > 1$ such that $A^n = \{0\}$.

Example 1.2.2 Consider the ring $\mathbb{Z}_{p^n}$, where p is a fixed prime and $n > 1$. $\mathbb{Z}_{p^n}$ has exactly one ideal for each positive divisor of p^n and no other ideals; these are simply the principal ideals $(p^k) = p^k\mathbb{Z}_{p^n}(0 \leq k \leq n)$. For $0 < k \leq n$, we have

$$(p^k)^n = (p^{kn}) = (0).$$

So that each proper ideal of $\mathbb{Z}_{p^n}$ is nilpotent.

Definition 1.2.6 (Nil ideal) A right (left, two sided) ideal A of a ring R is said to be a nil ideal if each element of A is nilpotent.

Remark 1.2.1 Every nilpotent ideal is a nil ideal but converse need not be true.

Example 1.2.3 Let p be a fixed prime and for each positive integer i let R_i be the ideals in $I/(p^{i+1})$, consisting of all nilpotent elements of $I/(p^{i+1})$. That is, consisting of the residue classes modulo P^{i+1} which contains multiple of p. Then $R_i^{i+1} = \{0\}$, where as $R_i^k = \{0\}$ for $k < i + 1$. Now consider the discrete direct sum of the rings R_i (i=1,2...). Since each element of T differs from zero in only a finite number of components that is, each element of T is nilpotent. Then T is a nil ideal in T but not a nilpotent ideal.

Remark 1.2.2 The sum of any finite number of nil (nilpotent) ideals of a ring R is again nil (nilpotent).

Definition 1.2.7 (Principal ideal) An ideal of a ring R generated by one element of R is called a principal ideal. The ideal generated by the element a of R is denoted by (a).

Example 1.2.4 Let $R[x]$ denote the ring of polynomials with real coefficients and let $< x^2 + 1 >$ denote the principal ideal generated by $x^2 + 1$. Then

$$< x^2 + 1 >= \{f(x)(x^2 + 1) \mid f(x) \in R[x]\}.$$

Remark 1.2.3 Let I be an ideal generated by a. Then I can be written in the following form

$$(a) = \{na + ra + as + \sum r_i a s_i \mid r, s, r_i, s_i \in R;\ n \in \mathbb{Z}\}$$

(i) If R is a commutative ring, then

$$(a) = \{\sum n_i x_i + \sum r_j y_j \mid n_i \in \mathbb{Z}, r_j \in R, x_i, y_j \in I\}.$$

(ii) If R is a commutative ring with unity, then

$$(a) = \{\sum r_i a s_i \mid r_i, s_i \in R\}.$$

Definition 1.2.8 (Maximal ideal) An ideal M in a ring R is said to be a maximal ideal provided that $M \neq R$ and whenever J is an ideal of R with $M \subset J \subset R$, then $J = R$.

Definition 1.2.9 (Minimal ideal) An ideal M in a ring R is called a minimal ideal if $M \neq 0$ and there exists no ideal I in R such that $\{0\} \subset I \subset M$.

Definition 1.2.10 (Socle) The socle of a ring R denoted by Soc(R) is the sum of the minimal left (right) ideals of R, if R has minimal left (right) ideals; otherwise Soc(R)= 0.

Definition 1.2.11 (Prime ideal) An ideal P in a ring R is said to be a prime ideal if it has the property that for any ideals A and B in R whenever $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Remark 1.2.4 Equivalently an ideal P in a ring R is prime if and only if any one of the following holds:

(i) If $a, b \in R$ such that $aRb \subseteq P$, then $a \in P$ or $b \in P$.

(ii) If (a) and (b) are principal ideals in R such that $(a)(b) \subseteq P$, then $a \in P$ or $b \in P$.

(iii) If U and V are left (right) ideals in R such that $U, V \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

Remark 1.2.5 If R is a commutative ring, then an ideal P of R is a prime ideal if and only if for all elements a and b in R, $ab \in P$ implies that $a \in P$ or $b \in P$.

Remark 1.2.6 In a commutative ring with identity every maximal ideal is a prime ideal.

Example 1.2.5 For an illustration of a ring possessing a non trivial prime ideal which is not maximal. Take $R = \mathbb{Z} \times \mathbb{Z}$, where the operations are performed componentwise. One can easily verify that $\mathbb{Z} \times \{0\}$ is a prime ideal of R. Since

$$\mathbb{Z} \times \{0\} \subset \mathbb{Z} \times \mathbb{Z}_e \subset R,$$

with $\mathbb{Z} \times \mathbb{Z}_e$ an ideal of R, where $\mathbb{Z}_e$ is the ring of even integers and hence $\mathbb{Z} \times \{0\}$ fails to be maximal.

Definition 1.2.12 (Semiprime ideal) An ideal P in a ring R is said to be a semiprime ideal in R if for every ideal I of R, $I^2 \subseteq P$ implies $I \subseteq P$.

Definition 1.2.13 (Direct sum and subdirect sum of rings) Let S_i, $i \in U$ be a family of rings indexed by the set U and S denote the set of all functions defined on the set U such that for each $i \in U$. The value of function at i is an element of S_i. If addition and multiplication in S is defined as: $(a+b)(i) = a(i) + b(i)$, for all $a, b \in S$, then S is a ring which is called the complete direct sum of rings S_i, $i \in U$. The set of all functions $i \in U$ is a subring of S which is called the discrete direct sum of rings S_i, $i \in U$. However, if U is a finite set, then the complete (discrete) direct sum of rings S_i, $i \in U$, as defined above is called a direct sum of rings S_i, $i \in U$.

Let T be a subring of the direct sum S of rings S_i and for each $i \in U$ let $\theta_i \in U$ be a homomorphism of S onto S_i defined by $a\theta_i = a(i)$, for $a \in S$. If $T\theta_i = S_i$ for every $i \in U$, then T is said to be a subdirect sum of the family of the rings S_i, $i \in U$.

Definition 1.2.14 (Radical ideal) An ideal I of a ring R is said to be a radical ideal of R if for $a \in R$ $a^n \in I$ for some integer $n \geq 1$, implies that $a \in I$.

Definition 1.2.15 (Jacobson Radical) The jacobson radical of a ring R, denoted by $rad\ R = \cap\{M \mid M$ is a maximal ideal of $R\}$.

Definition 1.2.16 (Prime Radical) The prime radical of a ring R, denoted by $\beta(R) = \cap\{P \mid P$ is a prime ideal of $R\}$.

Remark 1.2.7 If $\beta(R) = 0$, we say that the ring R is without prime radical or has zero prime ideal.

Example 1.2.6 The ring $F[x]$ of formal power series over a field F has zero prime radical.

Remark 1.2.8 If $rad\ R = 0$, then R is said to be a ring without jacobson radical.

Definition 1.2.17 (Nil(R)) Nil(R) is the unique maximal nil ideal of R.

Definition 1.2.18 (Prime ring) A ring R is said to be prime if $aRb = 0$ implies either $a = 0$ or $b = 0$.

Remark 1.2.9 A ring R is said to be prime if and only if the zero ideal (0) is a prime ideal in R.

Definition 1.2.19 (Semiprime ring) A ring R is said to be semiprime if $aRa = 0$, implies $a = 0$ for all $a \in R$.

Remark 1.2.10 A ring R is semiprime if and only if it has no nonzero nilpotent ideals.

Definition 1.2.20 (Center of a ring) The center of a ring R is the set of all those elements of R which commute with each element of R and is denoted by $Z(R)$ i.e,

$$Z(R) = \{x \in R \mid xr = rx \text{ for all r} \in \text{R}\}.$$

Remark 1.2.11 A ring R is commutative if and only if $Z(R) = R$.

Remark 1.2.12 Let R be a semiprime ring. Then

(i) The center of R contains no nonzero nilpotent elements.

(ii) The center of a nonzero one sided ideal in R is contained in the center of R. In particular, any commutative one sided ideal is contained in the center of R.

Definition 1.2.21 (Centralizer) Let S be a non-void subset of a ring R. Then the centralizer of S in R denoted by $C_R(S) = \{x \in R \mid xs = sx, \ for\ all\ s \in S\}$

Remark 1.2.13 For a prime ring R :

(i) The nonzero elements of $Z(R)$, the center of R are not zero divisors.

(ii) If d is a nonzero derivation of R, then d does not vanish on a nonzero left ideal of R.

(iii) If R contains a commutative nonzero left ideal (right ideal), then R is commutative.

(iv) If c and ac are in $Z(R)$ and c is not zero, then a is in $Z(R)$.

(v) In R, the centralizer of any nonzero one sided ideal is equal to $Z(R)$. In particular, if R has a nonzero central ideal, then R must be commutative.

Definition 1.2.22 (Simple ring) A ring R is said to be simple if it has no proper ideals.

Definition 1.2.23 (Semisimple ring) A ring R is said to be semisimple if its Jacobson radical is zero.

Definition 1.2.24 (Reduced ring) A ring R is said to be reduced if R has no nonzero nilpotent elements.

Definition 1.2.25 (Local ring) A ring R is said to be a local ring if it has a unique maximal ideal.

Definition 1.2.26 (Annihilator) Let R be a ring and S be a subset of R. Then $l_R(S) = \{x \in R \mid xS = 0\}$ is called left annihilator of S and $r_R(S) = \{x \in R \mid Sx = 0\}$ is called the right annihilator of S. $l_R(S)$ also denoted by $l(S)$ and $r_R(S)$ by $r(S)$.

Remark 1.2.14 $l(S)$ is a left ideal of R and $r(S)$ is a right ideal of R.

Definition 1.2.27 (ACC) A ring R is said to satisfy the ascending chain condition for ideals if, given any sequence of ideals $I_1, I_2, ...$ of R with

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ....,$$

there exists an integer n (depending on the sequence) such that $I_m = I_n$ for all $m \geq n$.

Definition 1.2.28 (Noetherian ring) A ring R is said to be noetherian ring if it satisfies ascending chain condition for ideals.

Definition 1.2.29 (DCC) A ring R is said to satisfy the descending chain condition for ideals if, given any sequence of ideals $I_1, I_2, ...$ of R with

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_n ...,$$

there exists an integer n such that

$$I_n = I_{n+1} = I_{n+2} = ...$$

Definition 1.2.30 (Artinian ring) A ring R is said to be artinian ring if it satisfies a descending chain condition for ideals.

Definition 1.2.31 (Twisted polynomial ring) Let K be a ring and σ be a ring endomorphism of K. Consider $K[x;\sigma] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in K,\ x \text{ is a variable}\}$, such that $xb \neq bx$ but $xb = \sigma(b)x$ for $b \in K$. Define addition as sum of the polynomials and multiplication as follows:

$$\left(\sum a_i x^i\right)\left(\sum b_j x^j\right) = \sum a_i \sigma^i(b_j) x^{i+j}.$$

Then $K[x;\sigma]$ is a ring and is known as Hilbert's twisted polynomial ring. $K[[x;\sigma]]$ is defined similarly and is called skew series ring.

Definition 1.2.32 (Principal left ideal domain) A principal left ideal domain (PLID) is a domain in which any left ideal is a principal ideal.

Definition 1.2.33 (Principal right ideal domain) A principal right ideal domain (PRID) is a domain R in which any right ideal is a principal ideal.

Definition 1.2.34 (Ore domain) A ring R is said to be a left (resp. right) Ore domain if it has no non zero divisors and the intersection of any two nonzero left (resp. right) ideals of R is not zero.

Definition 1.2.35 (Von Neumann regular ring) A ring R is said to von Neumann regular or simply regular if every $a \in R$ can be written in the form axa for some $x \in R$ (depending on a).

Definition 1.2.36 (Strongly regular ring) A ring R is said to be strongly regular if for any $a \in R$, there exists $x \in R$ such that $a = a^2x$.

Definition 1.2.37 (Goldie ring) A ring R is said to be a Goldie ring if

(i) R satisfies the ascending chain conditions on left annihilators.

(ii) Every independent set of left ideals of R is finite.

Definition 1.2.38 (Dense ideal) A right (resp. left) ideal J of a ring R is said to be a dense right (resp. dense left) ideal if for any $0 \neq r_1 \in R$, $r_2 \in R$ there exists $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in J$ (resp $rr_1 \neq 0; rr_2 \in J$). The collection of all dense right ideals is denoted by $\mathcal{D} = \mathcal{D}(R)$.

Definition 1.2.39 (Essential ideal) A right (left) ideal J of a ring R is said to be essential if for every nonzero right (left) ideal K of R, we have $J \cap K \neq 0$ and is denoted by $J \subseteq_e R$.

Definition 1.2.40 (Module) Let R be a ring. An additive abelian group M together with a function $R \times M \to M$ (defined as $(r, m) \mapsto rm$) is said to be a left module over R or a left R-module if for all $r, s \in R$ and $m_1, m_2 \in M$ the following conditions hold

(*i*) $r(m_1 + m_2) = rm_1 + rm_2$

(*ii*) $(r + s)m_1 = rm_1 + sm_1$

(*iii*) $r(sm_1) = (rs)m_1$

In case, if R has identity element I_R, then $I_R m = m$ holds for all $m \in M$. Such left module M is called a unital left R-module.

A unital right R-module is defined similarly via a function $M \times R \to M$ (defined as $(m, r) \mapsto mr$) and satisfying the obvious analogous of the above conditions.

Sometimes we denote a left R-module (resp. right R-module) by ${}_R M$ (resp. M_R)

Remark 1.2.15 For a commutative ring R, the notion of a left and a right module over R essentially coincide with each other and in this case we simply speak of a module over R.

Remark 1.2.16 If I is a left (right) ideal of a ring R, then I is a left (right) R-module. In particular $\{0\}$ and R are modules.

Definition 1.2.41 (Bimodule) Let R and S be arbitrary rings. An abelian group M is said to be a bimodule, more explicitly (R-S) module or bimodule ${}_R M_S$, if M

is both a left R-module and a right S-module and $r(ms) = (rm)s$ for all $m \in M$; $r \in R$ $s \in S$.

On the other hand if M is a left R-module and a left S-module, then M is a bimodule if the above condition is replaced by the condition

$$r(sm) = s(rm), \quad \text{for all } r \in R;\ s \in S \text{ and } m \in M.$$

Example 1.2.7 Let M_R be any right R-module and $E = Hom_R(M, M)$ its ring of endomorphism. Then it is readily verified that M turns out to be an E-module $_EM$ such that $e(mr) = (em)r$, for all $e \in E$ and $r \in R$. Thus M is a bimodule $_EM_R$.

Definition 1.2.42 (Submodule) A nonempty subset N of an R-module M is said to be an R-submodule (or simply a submodule) of M if

(i) $(N, +)$ is a subgroup of $(M, +)$.

(ii) for all $r \in R$ and $a \in N$, the module product $ra \in N$.

Definition 1.2.43 (Module homomorphism) Let M and N be two R-modules. A mapping $f : M \to N$ is called a module homomorphism or an R-homomorphism if

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$ for $m_1, m_2 \in M$.

(ii) $f(ra) = rf(a)$ for all $r \in R$ and $a \in M$.

Definition 1.2.44 (Faithful R-module) An R-module M is said to be faithful if its annihilator is zero.

Definition 1.2.45 (Simple module) A nonzero module M is simple if M has no proper non-zero submodules.

Definition 1.2.46 (Singular submodule) Let R be a ring. The singular submodule of an R-module M_R denoted by $\mathcal{Z}(M_R) = \{m \in M_R \mid r_R(m) \text{ is essential in } R_R\}$.

Remark 1.2.17

(*i*) If $M = R$, $\mathcal{Z}(R_R)$ is an ideal of R and is called the right singular ideal of R.

(*ii*) If $R \neq 0$, then $\mathcal{Z}(R_R) = R$.

(*iii*) M_R is a nonsingular module if $\mathcal{Z}(M_R) = 0$. In particular, R is a right nonsingular ring if $\mathcal{Z}(R_R) = 0$.

Definition 1.2.47 (Kasch ring) A ring R is said to be a right (left) Kasch ring if every simple right (left) R-module V can be embedded in R_R. R is called a Kasch ring if it is both a right and a left Kasch ring.

Definition 1.2.48 (Injective module) A right R-module I is said to be injective if for any monomorphism $g : A \to B$ of right R-module and any R-homomorphism $h : A \to I$, there exists an R-homomorphism $h' : B \to I$ such that $h = h' \circ g$.

$$\begin{tikzcd} & I & \\ O \arrow[r] & A \arrow[u, "h"] \arrow[r, "g"'] & B \arrow[ul, dashed, "h'"'] \end{tikzcd}$$

Definition 1.2.49 (Self injective rings) A ring R for which R_R is injective is called self injective ring.

Example 1.2.8 $\mathbb{Z}/n\mathbb{Z}$ $(n > 0)$ is a self injective ring.

Definition 1.2.50 (Primitive ring) A ring R is said to be primitive if R has a faithful simple module.

Definition 1.2.51 (Semiprimitive ring) A ring R is said to be semiprimitive if R has a faithful semisimple module.

Definition 1.2.52 (Primitive ideal) An ideal U in a ring R said to be a primitive ideal if the quotient ring R/U is primitive.

Definition 1.2.53 (Essential extension) A right R-module $E \supseteq M_R$ is said to be an essential extension of M if every nonzero submodule of E intersects M nontrivially.

Definition 1.2.54 (Maximal essential extension) An essential extension $E \supseteq M$ is said to be maximal if no module properly containing E can be an essential extension of M.

Definition 1.2.55 (Central extension) A ring T is called central extension of a ring R if $T = Z(T)R$.

Definition 1.2.56 (Rational extension) Let $N \subseteq M$ be right R-modules. M is said to be a rational extension of N if for any $y \in M$ and $x \in M\backslash\{0\}$, $x.y^{-1}N \neq 0$ i.e. there exists $r \in R$ such that $xr \neq 0$ and $yr \in N$.

Definition 1.2.57 (Injective hull) Let N be an extension of an R-module M. If N is a maximal essential extension of M, then N is called an injective hull or an injective envelope of M, denoted by $E_R(M)$ or $E(M)$.

Example 1.2.9 Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ be the additive group of rational numbers. Then $\mathbb{Q}_{\mathbb{Z}}$ is the injective hull of $\mathbb{Z}_{\mathbb{Z}}$.

Definition 1.2.58 (Rational hull) Let $I = E(M)$ and let $H = End(I_R)$, operating on the left of I. We define

$$\widetilde{E}(M) = \{i \in I : \forall\ h \in H,\ h(M) = 0 \Rightarrow h(i) = 0\}$$

$\widetilde{E}(M)$ is an R-submodule of I containing M and is called rational hull of M.

Definition 1.2.59 (Division Hull) For a domain A, a division ring D is called a division hull of A if there is a given inclusion map $A \hookrightarrow D$ such that D is generated as a division ring by A.

In other words there is no division ring D_0 such that $A \subsetneq D_0 \subsetneq D$.

Definition 1.2.60 (Free semigroup) Let A be a nonempty set. Let A^+ be the

set of all finite, nonempty words $a_1, a_2, ...a_m$ in the 'alphabet' A.
A binary operation is defined on A^+ by juxtaposition

$$(a_1a_2...a_m)(b_1b_2...b_n) = a_1a_2...a_mb_1b_2...b_n.$$

with respect to this operation, A^+ is a semigroup, called the free semigroup on A. The set A is generating set for A^+.

Definition 1.2.61 (Algebra) Let F be a field and A be a non-void set. A is said to be an algebra over F if

(*i*) A is a ring.

(*ii*) A is a vector space over F.

(*iii*) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all $x, y \in A$, $\alpha \in F$.

Definition 1.2.62 (Subdirect product) Let $\{A_i : i \in I\}$ be a collection of ideals of a ring R. Then R is said to be a subdirect product of $\{R/A_i : i \in I\}$ if the canonical-homomorphism $\psi : R \to \Pi_{i\in I}R/A_i$ is an injection.

Definition 1.2.63 (Prime spectrum) The prime spectrum of a ring R denoted by Spec(R) is the collection of all prime ideals of R, partially ordered under set inclusion.

Definition 1.2.64 (k-ring) Let k be a commutative ring with identity e. Then a k-ring A is a ring with identity e for which there exists a ring homomorphism $\sigma : k \to A$ (sending e to e).

Definition 1.2.65 (Free k-ring) Let k be any ring and $\{x_i : i \in I\}$ be a system of independent, noncommuting indeterminates over k. Then a "free k-ring" generated by $\{x_i : i \in I\}$ is denoted by $R = k < x_i : i \in I >$. The elements of R are polynomials in the noncommuting variables $\{x_i\}$ with coefficients from k. The coefficients are supposed to commute with each x_i. The "freeness" of R refers to the following universal property: if $\psi_0 : k \to k'$ is any ring homomorphism and $\{a_i : i \in I\}$ is any subset of k' such that each a_i commutes with each element of $\psi_0(k)$, then there exists a unique ring homomorphism $\psi : R \to k'$ such that $\psi \mid k = \psi_0$ and $\psi(x_i) = a_i$ for every $i \in I$.

Definition 1.2.66 (Balanced map) Let K be a commutative ring with unity and let V be a right K-module and let W be a left K-module. For P any additive abelian group, a map $\phi : V \times W \to P$ is said to be balanced if it is biadditive and satisfies $\phi(v\alpha, w) = \phi(v, \alpha w)$, $v \in V, w \in W, \alpha \in K$.

Definition 1.2.67 (Tensor product) An abelian group T is called a tensor product of V and W over K if the following properties hold.

(*i*) There is a balanced map $\tau : V \times W \to T$ such that T is additively generated by the image of τ.

(*ii*) Given any abelian group P and any balanced map $\rho : V \times W \to P$ there exists an additive map $\psi : T \to P$ such that $\rho = \tau\psi$. Tensor product of V and W over K is denoted by $V \otimes_K W$.

Definition 1.2.68 (Coproduct) Let A_1 and A_2 be algebras with unity 1 over a commutative ring K. Then a K-algebra A with 1 is a coproduct of A_1 and A_2 over K if:

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha"] \arrow[dr, "\sigma"'] & A \arrow[d, dashed, "\phi"] & A_2 \arrow[l, "\beta"'] \arrow[dl, "\tau"] \\
& P &
\end{tikzcd}$$

(*i*) There exist K-algebra homomorphisms $\alpha : A_1 \to A$ and $\beta : A_2 \to A$ such that $\alpha(A_1) \cup \beta(A_2)$ generates A as a K-algebra.

(*ii*) For any K-algebra P with 1 and homomorphisms $\sigma : A_1 \to P$ and $\tau : A_2 \to P$ there exists a homomorphism $\phi : A \to P$ such that $\alpha\phi = \sigma$ and $\beta\phi = \tau$, that is the diagram can always be completed.

Definition 1.2.69 (Derivation) Let R be a ring. An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ for all

$x, y \in R$.

Definition 1.2.70 (Uniform dimension) An R-module M_R is said to have uniform dimension n (written as $u.dim\ M = n$) if there is an essential submodule $V \subseteq_e M$ that is a direct sum of n uniform submodules. If on the other hand, no such an integer n exists, we write $u.dim\ M = \infty$.

1.3 Some well-known results

In this section we state some well known results which will be frequently used in the development of the subsequent chapters.

Theorem 1.3.1 (Zorn's Lemma) If $(S, \leq)$ is an inductive partially ordered set, then S has at least one maximal element.

Theorem 1.3.2 (Lam [43]) A right R-module M_R is injective if and only if it has no proper essential extension.

Theorem 1.3.3 (Lam [43]) Any right R-module M_R has a maximal essential extension.

Theorem 1.3.4 (Lam [43]) For any two modules M and I such that $M \subseteq I$, the following conditions are equivalent:

(i) I is maximal essential over M.

(ii) I is injective and is essential over M.

(iii) I is minimal injective over M.

Theorem 1.3.5 (Lam [43]) If M' be a submodule of M_R and $N \subseteq_d M$, then for any $f \in Hom_R(M', M)$, $f^{-1}(N) \subseteq_d M'$.

Theorem 1.3.6 (Lam [43]) Let M be a right R-module and N, N' and P are submodules of M. Then we have the following (*i*) If $N \subseteq_d M$, $N' \subseteq_d M$ then $N \cap N' \subseteq_d M$.

(*ii*) Let $N \subseteq P \subseteq M$. Then $N \subseteq_d M$ iff $N \subseteq_d P$ and $P \subseteq_d M$.

(*iii*) Assume M is a nonsingular module. Then $N \subseteq_d M$ iff $N \subseteq_e M$.

Theorem 1.3.7 (Lam [43]) Let M' be any submodule of I containing M. Then $M \subseteq_d M'$ iff $M' \subseteq \widetilde{E}(M)$, where $\widetilde{E}(M)$ is the rational hull of M.

Theorem 1.3.8 (Lam [43]) If a ring R has ACC on annihilator ideals, then an ideal U is dense in R iff it contains a non-zero divisor.

Theorem 1.3.9 (Beidar [5]) Let J be a dense right ideal of a ring R. Then J is an essential right ideal of R.

Theorem 1.3.10 (Beidar [5]) Let I be a 2-sided ideal of a ring R. Then the following conditions are equivalent:

(*i*) $l(I) = 0$.

(*ii*) I is a dense right ideal of R.

(*iii*) I is an essential right ideal.

(*iv*) I is essential as a 2-sided ideal (that is for any ideal $J \neq 0$, $I \cap J \neq 0$).

Theorem 1.3.11 (Beidar [5]) Let I be a 2-sided ideal of R. Then

(*i*) $l(I) = r(I)$

(*ii*) $l(I) \cap I = 0$

(*iii*) $I + l(I)$ is a dense right ideal of R.

Theorem 1.3.12 (Rowen [63]) Suppose $R \in$C$-\mathcal{A}lg$ (category of algebras over a commutative ring C), $x_1, x_2 ... x_t$ are arbitrary elements of R and $T : \sum_{i=1}^{t} Cx_i \rightarrow \sum_{i=1}^{t} Cx_i$ is a map (in C-Mod). Viewing T as an image of a $t \times t$ matrix T', we have

the following formula for any t-normal polynomial $f(x_1, x_2, ...x_k)$:
$(trT')f(x_1, ...x_k) = \sum_{i=1}^{t} f(x_1, ...x_{i-1}, Tx_i, x_{i+1}, ...x_k)$

Theorem 1.3.13 (Rowen [63]) Suppose D is a division algebra over an algebraically closed field F and D has a (possibly infinite) base over F of cardinality$<\mid F \mid -1$. Then $D = F$. (Moreover, if F is not algebraically closed, then D is algebraic over F.

Theorem 1.3.14 (Rowen [63]) Given a ring R, let I be an infinite set of cardinality $\geq\mid R \mid$, and put $R' = R^I$, writing (r_i) for the element of R' whose i-component is $r_i \in R$ for all $i \in I$. Identify R as a subring of R' under the "diagonal injection "$r \to (r_i)$, with each $r_i = r$. Then $N(R) = Nil(R') \cap R$. Consequently, $N(R)$ is a proper ideal of R and there is an injection $R/N(R) \to R'/Nil(R')$.

Theorem 1.3.15 (Rowen [63]) Let R be a semiprimitive ring. If R has no nonzero nil ideals, then $R[\lambda]$ is semiprimitive, where $R[\lambda]$ is the polynomial ring over R on the commuting indeterminates λ_i, $i \in I$.

Theorem 1.3.16 (Rowen [63]) If R is a prime PI-ring and $S = Z(R)\backslash\{0\}$, then $S^{-1}Z(R) \subseteq Z(S^{-1}R)$, equality holding if S is regular.

Theorem 1.3.17 (Rowen [63]) The following conditions are equivalent for a ring R:

(i) R is a finite direct product of simple artinian rings.

(ii) $R = Soc(R)$

(iii) R is semiprime and left artinian.

Theorem 1.3.18 (Beidar [5]) Let L be a minimal left ideal of a ring A. Suppose that $L^2 \neq 0$. Then there exists an idempotent $e \in L$ such that $L = Ae$. Moreover eAe is a division ring. Further, if A is semiprime ring and $v \in A$ is an idempotent such that vAv is a division ring, then Av is a minimal left ideal of A.

Theorem 1.3.19 (Beidar [5]) Let R be a primitive ring with extended centroid C. Then R is GPI if and only if R contains a minimal idempotent e such that $dim_C(eRe) < \infty$.

Theorem 1.3.20 (Beidar [5]) Let R be a primitive ring and let V be any faithful irreducible right R-module with associated division ring D. Then

(i) $Soc(R) = \{r \in R \mid rank\ r < \infty\}$.

(ii) $Soc(R) = Soc(H)$ where H is any ring such that $R \subseteq H \subseteq Q_S(R)$ and $Q_S(R)$ is symmetric ring of quotients of R.

CHAPTER II

Classical rings of quotients and embedding theorems

2.1 Introduction

The present chapter is a general introduction to the theory of rings of quotients (rings of fractions), in setting of noncommuting rings.

In section 2.2 we discussed the general issues of inverting a given multiplicative set S of a nonzero elements in a (possibly) noncommuting ring R. If R is a domain and $S = R\backslash\{0\}$, a related issue is that of embedding R into a division ring. Unfortunately such embeddings need not always exist, even if such embeddings exist, they may not be unique, as shown by an intriguing example of J.L Fisher.

Section 2.3 deals with the study of Ore's localization theory developed by Ore in the early 1930's. Here we find the necessary and sufficient condition for constructing the (Ore) localization RS^{-1} with respect to a given multiplicative set $S \subseteq R$. Letting S be the multiplicative set of all nonzero divisors in R, in particular, we arrive at the notion of right Ore rings, which are rings with a classical (total) ring of quotients. Finally, we discuss $Q^r_{cl}(R)$, the right classical ring of quotients and $Q^l_{cl}(R)$, the left classical ring of quotients for particular choices of the multiplicative set S of the ring R.

2.2 Non commutative localizations

For any commutative domain R, we can formally invert the non zero elements of R to form a unique quotient field (or field of fractions) for R. In commutative algebra, the general procedure of localizing any commutative ring R at a multiplicative set S yields a commutative ring R_S and a ring homomorphism $\epsilon : R \to R_S$ such that $\epsilon(s)$ is a unit in R_S for every $s \in S$, and ϵ is universal with respect to this property. Moreover, we have the following two key facts for ϵ and R_S.

(i) Every element in R_S has the form $\epsilon(r)\epsilon(s)^{-1}$. (2.2.1)

(ii) $Ker\epsilon = \{r \in R/rs = 0 \text{ for some } s \in S\}$. (2.2.2)

The ring R_S is called the localization of R at S. To simplify the notation, we write the elements of R_S as r/s or rs^{-1} (instead of $\epsilon(r)\epsilon(s)^{-1}$). We add fractions by taking common denominators and multiply fractions by multiplying numerators and denominators. The classical case of embedding a commutative domain R into its quotient field corresponds to the localization of R at the multiplicative set $R\backslash\{0\}$.

In commutative algebra, localization provides one of the most powerful tools for proving theorems. Thus, in studying noncommutative rings, it is natural to ask first how much of localization machinery can be made to work in the noncommutative setting. In studying the theory of noncommutative localization for any multiplicative set S in any ring R, we can define a universal S-inverting ring R_S. But we lose both of the properties (2.2.1) and (2.2.2). This generally compromises the usefulness of R_S. Also Mal'cev [50] has shown (Theorem 2.2.1) that a noncommutative domain can not be embedded in any division ring even if such embeddings exist, they may not be unique as shown by the following intriguing example of J.L Fisher [16].

Example 2.2.1 Consider a domain $A = \mathbb{Q} < u, v >$. Then there exist embeddings $\epsilon_n : A \to D_n$ where D_n's are minimal division rings over $\epsilon_n(A)$ for $n \geq 2$, but there is no isomorphism (or homomorphism) $f : D_m \to D_n$ for $m \neq n$ such that $f \circ \epsilon_m = \epsilon_n$. The free algebra $\mathbb{Q} < u, v >$, therefore has infinitely many essentially different "division rings of fractions".

We begin with the following results proved by Mal'cev [50]

Lemma 2.2.1 Let a, b, c, d, x, y, u, v be elements of a semigroup H. If H is embeddable into a group G, then $ax = by$, $cx = dy$, $au = bv \Rightarrow cu = dv$ in H.

Proof Working in the group G, we have $b^{-1}a = yx^{-1} = d^{-1}c$ from the first two equations and $b^{-1}a = vu^{-1}$ from the third equation. Therefore, $d^{-1}c = vu^{-1} \in G$ and hence $cu = dv \in H$. (Alternatively, as suggested by D. Moulton, $cu = cx \,.\, x^{-1}a^{-1} \,.\, au = dy \,.\, y^{-1}b^{-1} \,.\, bv = dv$ in G.)

Proposition 2.2.1 There exists a cancellative semigroup H with elements a, b, c, d, x, y, u, v such that $ax = by$, $cx = dy$, $au = bv$ but $cu \neq dv$. In particular, H can not be embedded in any group G.

Proof Let $\overline{H}$ be the free semigroup on the letters A, B, C, D, X, Y, U, V. For two words W and W', let us define $W \sim W'$ if W can be transformed into W' by a finite number of replacements of subwords of length two of the following kinds.

$$AX \longleftrightarrow BY, \quad CX \longleftrightarrow DY, \quad AU \longleftrightarrow BV. \tag{2.2.3}$$

Clearly '$\sim$' is an equivalance relation on words. Let H be the set of $\sim$-equivalence classes and let w denotes the class of word $W \in \overline{H}$. The multiplication in $\overline{H}$ induces a multiplication in H that makes H into a semigroup. The classes $a, b, \cdots \in H$ of $A, B, \cdots$ now satisfy $ax = by$, $cx = dy$ and $au = bv$. But we don't have $cu = dv \in H$. Since the word CU simply cannot be transformed into DV. The only thing that remains to be verified is the fact that H does satisfy both of the cancellation laws.

Let us say that a word $\overline{H}$ is reduced if it does not contain a subword AX, CX or AU. Using the forward transformations in (2.2.3), it is clear that any word $W \in \overline{H}$ is $\sim$-equivalent to a unique reduced word. Equipped with this knowledge, let us now prove the left cancellation law:

$$ww_1 = ww_2 \in H \Rightarrow w_1 = w_2 \in H.$$

We may assume that w, w_1 and w_2 are classes of reduced words W, W_1 and W_2. If WW_1 and WW_2 are both reduced, then have $WW_1 = WW_2$; hence $W_1 = W_2 \in \overline{H}$ and $w_1 = w_2 \in H$.

Now assume, WW_1 is not reduced. Let us examine a typical case, say, $W = \cdots LA$, $W_1 = XM_1N_1 \cdots$. In this case, the class ww_1 is represented by the reduced word $\cdots LBYM_1N_1 \cdots$. If W_2 did not start with X or U, then WW_2 is already reduced and it is not $\cdots LBYM_1N_1 \cdots$, which contradicts $ww_1 = ww_2$. If W_2 starts with U, then ww_2 is given by a reduced word of the form $\cdots LBV \cdots$, still contradicting $ww_1 = ww_2$. Thus, we must have $W_2 = XM_2N_2 \cdots$, so that ww_2 is given by the reduced word $\cdots LBYM_2N_2\cdots$. But then we must have $M_1N_1\cdots = M_2N_2\cdots$, which implies that $W_1 = W_2 \in \overline{H}$ and hence $w_1 = w_2 \in H$. The other cases are similarly dealt with and the right cancellation law can be proved in the same manner.

Theorem 2.2.1 Let R be the semigroup algebra kH, where H is as in Proposition 2.2.1, and k is any domain. Then R is a domain and R cannot be embedded into any division ring.

Proof It suffices to prove that R is a domain. Suppose that there is an equation

$$(\sum_i \alpha_i w_i)(\sum_j \alpha'_j w'_j) = 0 \in R \tag{2.2.4}$$

where $\alpha_i \neq 0 \neq \alpha'_j$ and the w_i's (resp. w_j's) are given by different reduced words W_i's (resp W_j's). Note that the length of an element in H is well defined, since the transformations allowed in (2.2.3) are all length preserving. We may, therefore, assume that words W_i (resp. W'_j) have the same length. (If otherwise, we just replace $\sum_i \alpha_i w_i$ by the subsum given by the terms of longest length and do the same for $\sum_j \alpha_j w'_j$). In order to cancel out the class $w_1 w'_1$, we must have $w_1 w'_1 = w_i w'_j$ for some $i \neq 1,\ j \neq 1$.

Since $W_1 \neq W_i$ and they have the same length, the only way for $w_1 w'_1 = w_i w'_j$ to be possible is that we have

$$W_1 = K\cdots LA, \quad W'_1 = XMN\cdots$$

$$W_i = K\cdots LB, \quad W'_j = YMN\cdots$$

But then on the L.H.S of (2.2.4) above, we have a term $\alpha_1\alpha'_j w_1 w'_j$ corresponding to the reduced word $K\cdots LAYMN\cdots$ which clearly cannot be cancelled out by any other term, a contradiction.

Proposition 2.2.2 If S is a multiplicative subset of a ring R, then there exists an S inverting homomorphism ϵ from R to some ring, denoted by R_S, with the following universal property:

For any S-inverting homomorphism $\alpha : R \to R'$, R' a ring there exists a unique ring homomorphism $f : R_S \to R'$ such that $\alpha = f \circ \epsilon$.

Proof Fix a presentation of R by generators and relations. For each $s \in S$, adjoin a new generator s^* and two additional relations $\bar{s}s^* = 1$, $s^*\bar{s} = 1$, where $\bar{s}$ is an element in the free $\mathbb{Z}$-algebra that maps to s in the given presentation. The new set of generators and relations defines a ring R_S, along with a ring homomorphism $\epsilon : R \to R_S$. For each $s \in S$ the image of s^* in R_S provides an inverse for $\epsilon(s)$, so $\epsilon(S) \subseteq U(R_S)$. The asserted universal property of ϵ follows quickly from the definition of R_S.

Remark 2.2.1 Contrary to the commutative case, the universal S inverting ring R_S may be the zero ring, even though $R \neq 0$ and $0 \notin S$.

Example 2.2.2 Let $R = \mathbb{M}_n(k)$ $(n \geq 2)$, where k is a non zero ring, and let S be the multiplicative set $\{1, E_{11}\}$ where E_{ij} denote the matrix units. Being an ideal in R, the kernel of $\epsilon : R \to R_S$ has the form $M_n(U)$, where U is an ideal in k (cf Theorem 3.1, [41]). But $E_{11}E_{22} = 0$ implies that $E_{22} \in Ker\epsilon$, so we have $1 \in U$, i.e. $U = k$. Therefore, ϵ is the zero map and $R_S = (0)$ (Here, R is not a domain. But even when R is a domain, R_S may still be equal to (0)).

Every domain can not be embedded in a division ring. There are some special classes of domains that have been proved to be embeddable. For instance we have the following nice results due to Lam [43].

Theorem 2.2.2 Any right noetherian domain can be embedded in a division ring. In particular, any PRID (principal right ideal domain) can be embedded in a division ring.

Proof can be followed by Theorem 2.4.1.

Theorem 2.2.3 A domain A has a division hull iff A can be embedded in a division ring.

Remark 2.2.2

(i) Two division hulls of a domain A are regarded as the same if they are isomorphic over A.

(ii) A has infinitely many mutually different division hulls.

Clearly a domain $A = C < u, v >$, the free algebra on two generators u, v over any field C. It is by no means clear that $C < u, v >$ can be embedded in a division ring. Since $C < u, v >$ is neither left nor right noetherian, Theorem 2.2.2 does not apply directly. In order to get embeddings of $C < u, v >$ into division rings, we shall make crucial use of Hilbert's skew polynomial rings. We know that for any ring K equipped with an endomorphism σ, the skew polynomial ring $K[x; \sigma]$ consists of left polynomials of the form $\sum a_i x^i$ ($a_i \in K$) which are multiplied using Hilbert's twist $xa = \sigma(a)x$ (for every $a \in K$). The following basic fact will prove.

Lemma 2.2.2 Let $\sigma : K \to K$ be an injective endomorphism of the ring K and let $R = K[x; \sigma]$. If $\{t_i : i \in I\} \subseteq K$ are right linearly independent over $\sigma(K)$. Then $\{t_i x : i \in I\} \subseteq R_R$ are right linearly independent over R.

Proof Suppose $\sum_i (t_i x) f_i = 0$ where $f_i \in R$ are almost all zero. Write $f_i = \sum_i a_{ij} x^j$ $(a_{ij} \in K)$. Then

$$0 = \sum_i (t_i x) \sum_j a_{ij} x^j = \sum_j (\sum_i t_i \sigma(a_{ij})) x^{j+1}.$$

Therefore for each j, we have $\sum_i t_i \sigma(a_{ij}) = 0$, and so $\sigma(a_{ij}) = 0$ for all i, j. Since σ is injective, it follows that $f_i = \sum_j a_{ij} x^j = 0$ for all i.

If K is a division ring, then any endomorphism $\sigma : K \to K$ is automatically injective, from which one can see easily that $K[x; \sigma]$ is a domain. In addition, the usual Euclidean algorithm argument can be used to show that $K[x; \sigma]$ is a PLID. Therefore, we have from Theorem 2.2.2:

Corollary 2.2.1 If σ is an endomorphism of a division ring K, then $K[x; \sigma]$ can be embedded in a division ring.

Therefore to embed $C < u, v >$ into a division ring, we might try to embed it first into $K[x; \sigma]$ where K is a division ring (or even a field). This will be accomplished with the help of the following observations.

Lemma 2.2.3 [Jategaonkar's Lemma] Suppose a, b are two elements in a ring R that are right linearly independent over R. Let $C \subseteq R$ be any nonzero subring whose elements commute with a and b. Then the subring of R generated by a, b over C is a free C-ring on a, b.

Proof If a, b are not free over C, choose a nonconstant polynomial $f(x,y) \in C < x, y >$ of the least degree n such that $f(a,b) = 0$. Express f in the form $\alpha + xg(x,y) + yh(x,y)$ $(\alpha \in C)$, where $g(x,y) \neq 0$. From

$$0 = f(a,b)b = a(g(a,b)b) + b(\alpha + h(a,b)b,) \qquad (2.2.4)$$

we see that $g(a,b)b = 0$. Now write g in the form $\beta + xp(x,y) + yq(x,y)$ $(\beta \in C)$. Then we have

$$deg\ g \leq n-1, \quad deg\ p \leq n-2, \quad deg\ q \leq n-2, \qquad (2.2.5)$$

and

$$0 = g(a,b)b = a(p(a,b)b) + b(\beta + q(a,b)b). \qquad (2.2.6)$$

The latter implies that $p(x,y)y$ and $\beta + q(x,y)y$ are both satisfied by a, b. Using (2.2.5), we see that $p(x,y) = q(x,y) = 0$ and $\beta = 0$, contradicting $g(x,y) \neq 0$.

Theorem 2.2.4 For $n \neq m$ (both > 1), there does not exist a ring homomorphism $f : D_m \to D_n$ such that $f \circ \epsilon_m = \epsilon_n$ (so that D_n and D_m give essentially different division hulls of $C < u, v >$).

Proof Suppose f exists. Applying it to equation

$$\left(\epsilon_m(u)^{-1}\epsilon_m(v)\right)^m = \epsilon_m(v)\epsilon_m(u)^{-1} \in D_m \qquad (2.2.7)$$

we get

$$\left(\epsilon_n(u)^{-1}\epsilon_n(v)\right)^m = \epsilon_n(v)\epsilon_n(u)^{-1} = \left(\epsilon_n(u)^{-1}\epsilon_n(v)\right)^n \in D_n. \qquad (2.2.8)$$

Since $\epsilon_n(u)^{-1}\epsilon_n(v) = x^{-1}tx$ in D_n, (2.2.8) gives $t^{n-m} = 1 \in D_n$, a contradiction.

The embeddability of $C < u, v >$ in a division ring is an important fact, even though there is no uniqueness in such an embedding. Recalling that any free algebra $C < X >$ with X countable can be embedded in $C < u, v >$ (cf Example 1.2, [41]), we see that such $C < X >$ can be embedded in a division ring. By a different method, Lam [43] have in fact shown that $C < X >$ can be embedded in a division ring, for any set X and any division ring C.

The following theorem is due to Robinson [60].

Theorem 2.2.5 If a domain R can be embedded in a direct product of division rings D_i $(i \in I)$, then R can be embedded in a division ring.

For developing the proof of the theorem we need the following result that can be found in [41].

Lemma 2.2.4 A ring R is strongly regular iff it is von Neumann regular and reduced. Such a ring is always a subdirect product of division rings.

Proof of the Theorem 2.2.5 Let $P = \prod_i D_i$ and write each element $x \in P$ in the form $(x_i)_{i \in I}$. For such $x \in P$, we define an element $x^* = (x_i^*)_{i \in I} \in P$ by $x_i^* = 0$ if $x_i = 0$ and $x_i^* = x_i^{-1}$ if $x_i \neq 0$. Also we define

$$U_x = \{(a_i)_{i \in I} \mid \forall i \in I,\ x_i \neq 0 \Longrightarrow a_i = 0\} \qquad (2.2.9)\}$$

Clearly, U_x is an ideal in P (infact, $U_x = l_R(x) = r_R(x)$) and we have

$$1 - xx^* \in U_x \qquad (2.2.10),$$

$$(U_x + U_y + \dots + U_z)xy\dots z = 0 \qquad (2.2.11)$$

viewing R as a subring of P, let $U = \sum U_x$, where x ranges over $R \backslash \{0\}$. Then $U \neq P$, for otherwise $1 \in U_x + U_y + \dots + U_z$ for suitable $x, y, z \dots, \in R \backslash \{0\}$ and (2.2.11) would give $xy \dots z = 0$, contradicting the fact that R is a domain. Since each D_i is strongly regular so are $P = \prod_i D_i$ and P/U and hence there exists a ring homomorphism f from P to a suitable division ring D, with $f(U) = 0$. Now for any

$x \in R\backslash\{0\}$, (2.2.10) implies that

$$0 = f(1 - xx^*) = 1 - f(x)f(x^*), \quad \text{so } f(x) \neq 0 \in D.$$

Therefore $f \mid R$ gives the desired embedding of R into a division ring.

Corollary 2.2.2 If a domain R can be embedded in a strongly regular ring R', then R can be embedded in a division ring.

Proof By Lemma 2.2.4, R' is a subdirect product of a family of division rings D_i ($i \in I$). In particular R' (and hence R) can be embedded into $\prod_{i \in I} D_i$. Since R is a domain, Theorem 2.2.5 implies that R can be embedded in a division ring.

2.3 Ore localizations

In this section we present a general introduction to the theory of rings of quotients in the setting of non commutative rings. We continue to write S for a multiplicative set in a ring R, so we have $S.S \subseteq S$, $1 \in S$ and $0 \notin S$. The ring R_S receiving the universal S-inverting homomorphism $\epsilon : R \to R_S$ is too difficult to work with, since elements of R_S which are sums of words in $\epsilon(r)$ and $\epsilon(s)^{-1}$ have very complicated forms like $\epsilon(r)\epsilon(s)^{-1} + \epsilon(s') + \epsilon(r'')\epsilon(s'')^{-1}$, where $r, r', r'' \in R$ and $s, s', s'' \in S$ and we have little control over $\mathrm{Ker}\epsilon$. Introducing some additional conditions on S we can form simpler classical ring of fractions. The following definition due to Lam [43] sets forth the features of the kind of classical ring of fractions we would like to form.

Definition 2.3.1 (Ring of quotients or ring of fractions) A ring R' is said to be a right ring of quotients or right ring of fractions (with respect to multiplicative set $S \subseteq R$) if there is given ring homomorphism $\psi : R \to R'$ such that

(a) ψ is S-inverting

(b) Every element of R' has the form $\psi(a)\psi(s)^{-1}$ for some $a \in R,\ s \in S$.

(c) $Ker\psi = \{r \in R : \ rs = 0 \text{ for some } s \in S\}$

Analogously we can define a left ring of quotients or left ring of fractions.

Remark 2.3.1 Contrary to the situation with R_S (cf Example 2.2.1) we have always $R' \neq 0$ here in view of (c)

Definition 2.3.2 (Right permutable or Right Ore set) A multiplicative set $S \subseteq R$ is said to be right permutable if for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$. S is also known as right Ore set.

Definition 2.3.3 (Right reversible) S is said to be right (left) reversible if for $a \in R, s'a = 0$ $(as' = 0)$ for some $s' \in S$, then $as = 0$ $(sa = 0)$.

Definition 2.3.4 (Right denominator) If the multiplicative set $S(\subseteq R)$ is both right permutable and right reversible, then S is called a right denominator set.

Definition 2.3.5 (Divisor set) Let R be any ring and $S \subseteq R$, a multiplicative set of regular elements. Then S is said to be a right (left) divisor set if S is a right (left) permutable set.

We come now to the first major result in this section, which is due to Ore, Asano and others. Ore started the investigation of noncommutative localization in the early 1930s by proving the theorem below for R a domain and $S = R\backslash\{0\}$. Asano and others extended Ore's theory to more general rings.

Theorem 2.3.1 The ring R has a right ring of quotients with respect to S if and only if S is a right denominator set.

Proof Let R has a right ring of quotients R_S wrt S . Then by Definition 2.3.1(a) ψ is S-inverting. Write $\psi(s)^{-1}\psi(a) = \psi(r)\psi(s')^{-1}$ for $r \in R$; $s' \in S$. Multiplying both side by $\psi(s)$ from left $\psi(s)\psi(s)^{-1}\psi(a) = \psi(s)\psi(r)\psi(s')^{-1}$we get $\psi(a) = \psi(s)\psi(r)\psi(s')^{-1}$ i.e $\psi(a)\psi(s') = \psi(s)\psi(r)$ then we have $\psi(as') = \psi(sr)$ $\Rightarrow \psi(as') - \psi(sr) = 0 \Rightarrow \psi(as' - sr) = 0$ i.e $as' - sr \in Ker\psi$ this implies that $(as' - sr)s'' = 0$ for some $s'' \in S \Rightarrow as's'' - srs'' = 0 \Rightarrow as's'' = srs''$ but $as's''$ is an element of aS and srs'' is an element of $sR \Rightarrow as's'' - srs'' \in aS \cap sR \Rightarrow aS \cap sR \neq \emptyset$ $\Rightarrow S$ is right permutable.

Let $s'a = 0$ for $a \in R$ and $s' \in S$. Then $\psi(s'a) = \psi(0)$ i.e $\psi(s')\psi(a) = 0$ since $\psi(s')$ is invertible in R_S, so it follows that $\psi(a) = 0$. Thus $a \in Ker\psi$. By Definition 2.3.1(c) $as = 0$ for some $s \in S$, showing that S is right reversible. Hence S is a right denominator set.

Conversely assume that S is a right denominator set and construct a right ring of quotients denoted by RS^{-1}. Since elements of RS^{-1} will be right fractions of the form $as^{-1}(a \in R,\ s \in S)$, we start the construction by working with $R \times S$. We define a relation $\sim$ on $R \times S$ as follows:

$$\begin{aligned} &(a, s) \sim (a', s') \text{ (in } R \times S) \text{ iff there exist} \\ &b, b' \in R \text{ such that } sb = s'b' \text{ and } ab = a'b' \in R \end{aligned} \tag{2.3.1}$$

Intuitively, the condition means that after we *blow up* s and s' to the common denominator $sb = s'b' \in S$, the numerators ab and $a'b'$ are the same. Notice that although $sb = s'b' \in S$, b and b' themselves need not belong to S.

We claim that $\sim$ is an equivalence relations on $R \times S$. Reflexivity and symmetry need no verification, so let us just prove transitivity. Assume that $(a, s) \sim (a', s')$ as in (2.3.1) and also that $(a', s') \sim (a'', s'')$, so that we have $c, c' \in R$ with $s'c = s''c' \in S$ and $a'c = a''c' \in R$. From $(s'c)S \cap (s'b')R \neq \emptyset$, there exist $r \in R$ and $t \in S$ such that $s'b'r = s'ct \in S$. Using right reversibility, we have $b'rt' = ctt'$ for some $t' \in S$. Now

$$\begin{aligned} &sbr = s'b'r = s''c't \in S \\ &\Rightarrow s(brt') = s''(c'tt') \in S \\ &\Rightarrow a(brt') = a'b'rt' = a'ctt' = a''(c'tt'). \end{aligned}$$

So we have checked that $(a, s) \sim (a'', s'')$. In (2.3.1), if we let $b' = 1$, we see that $(a, s) \sim (ab, sb)$ as long as $sb \in S$. therefore, we can think of $\sim$ as the best equivalence relation which identifies (a, s) with (ab, sb) $(\forall a \in R,\ s \in S,\ sb \in S)$. This remark enables us to work with $\sim$ very efficiently. We need a notation for the equivalence class of (a, s). In anticipation of our goal, we write a/s or as^{-1} for this equivalence class. The set of all equivalence classes will be denoted by RS^{-1}; of course as^{-1} is so for only a formal expression in RS^{-1}. To define addition in RS^{-1}, we observe that any two fractions a_1/s_1, a_2/s_2 can be brought to a common denominator. More formally, from $s_1S \cap s_2R \neq \emptyset$, we get elements $r \in R$, $s \in S$ such that

$s_2r = s_1s \in S$, so now $a_1/s_1 = a_1s/s_1s$ and $a_2/s_2 = a_2r/s_2r$. We can then define

$$a_1/s_1 + a_2/s_2 = (a_1s + a_2r)/t \quad \text{where } t = s_1s = s_2r. \tag{2.3.2}$$

After showing that this is a well defined binary operation on RS^{-1}, one can go ahead to show that $(RS^{-1}, +)$ is an additive group, with zero element $o/1$. We shall not present the details here, but note quickly that $\psi(a) = a/1$ gives a group homomorphism $\psi : R \to RS^{-1}$ with

$$Ker\psi = \{a \in R : \ (a,1) \sim (0,1)\} = \{a \in R : \ as = 0 \text{ for some } s \in S\}, \tag{2.3.3}$$

We also note in passing that, in connection with (2.3.2), any finite number of fractions can be bought to a common denominator, by using the permutability property together with induction. So far we have used the permutability condition (Definition 2.3.2) only in the case when both a and s are in S. We shall need the full version of Definition 2.3.2 in the next step, when we try to define multiplication on RS^{-1}. In order to multiply a_1/s_1 with a_2/s_2, we use $s_1R \cap a_2S \neq \emptyset$ to find $r \in R$. and $s \in S$ such that $s_1r = a_2s$. Then we define

$$(a_1/s_1)(a_2/s_2) = (a_1r)/(s_2s) \tag{2.3.4}$$

keeping in mind that $(a_1s_1^{-1})(a_2{s_2}^{-1})$ should be

$$a_1(s_1^{-1}a_2){s_2}^{-1} = a_1(rs^{-1}){s_2}^{-1} = a_1r(s_2s)^{-1}$$

again, one can check that (2.3.4) gives a well defined multiplication on RS^{-1} and finally that $(RS^{-1}, +, \times)$ is a ring. Note that $1/1$ is the multiplicative identity in RS^{-1} and that the map ψ defined just before (2.3.3) is clearly a ring homomorphism from R to RS^{-1}. Also $1/s$ $(s \in S)$ is the inverse of $\psi(s) = s/1$, so ψ is S inverting. Finally, we see easily that $a/s = \psi(a)\psi(s)^{-1}$. By (2.3.3) we have now shown that RS^{-1} is a right ring of fractions of R with respect to S, completing the proof.

Corollary 2.3.1 If S is a right denominator set, then $\psi : R \to RS^{-1}$ is a universal S-inverting homomorphism. In particular, there is a unique isomorphism $g : R_S \to RS^{-1}$ such that $g \circ \epsilon = \psi$, where $\epsilon : R \to R_S$.

Proof It suffices to prove the first statement. Let $\alpha : R \to T$ be any S-inverting homomorphism. We define $f : RS^{-1} \to T$ by

$$f(a/s) = \alpha(a)\alpha(s)^{-1} \qquad (a \in R,\ s \in S). \tag{2.3.5}$$

If $b \in R$ is such that $sb \in S$, then $\alpha(s)\alpha(b) = \alpha(sb)$ is a unit in T, so $\alpha(b)$ is also a unit in T, but then $\alpha(ab)\alpha(sb)^{-1} = \alpha(a)\alpha(b)\alpha(b)^{-1}\alpha(s)^{-1} = \alpha(a)\alpha(s)^{-1}$. This shows that $f : RS^{-1} \to T$ is well defined. From (2.3.2) and (2.3.5), we can show easily that f is a ring homomorphism, with $f \circ \psi = \alpha$. Finally, f as defined in (2.3.5) is clearly the only homomorphism from RS^{-1} to T satisfying $f \circ \psi = \alpha$, since $a/s = \psi(a)\psi(s)^{-1} \in RS^{-1}$.

Of course, we also have the notion of left permutability, left reversibility and left denominator set. If S is a left denominator set, then we can obtain a left ring of quotients or left ring of fractions of R with respect to S, denoted by $S^{-1}R$. From Corollary 2.3.1 and its corresponding left version, we deduce the following result:

Corollary 2.3.2 If both RS^{-1} and $S^{-1}R$ exist, then $RS^{-1} \cong S^{-1}R$ ($\cong R_S$) over R.

2.4 Right Ore rings and domains

To begin this section, let us consider some particular choices of the multiplicative set $S \subseteq R$.

I If S is central in R, then S is clearly both a left and right denominator set and we can safely identify $S^{-1}R$ with RS^{-1}. We speak of $S^{-1}R = RS^{-1}$ as a "Central localization" of R. In this case, we have infact $RS^{-1} \cong R \otimes_C CS^{-1}$, where C is center of R.

II If S consists only of regular elements of R, then S is clearly left and right reversible.

III Let S be the multiplicative set of all regular elements. We say that R is a right Ore ring if and only if S is right-permutable, if and only if RS^{-1} exists by virtue of II. In this case we speak of RS^{-1} as the (total) classical right ring of quotients of R and denote it by $Q^r_{cl}(R)$. The left analogues of these notions are defined similarly. If R is both left and right Ore, then we shall say that R is an Ore ring, in this case $Q^r_{cl}(R) = Q^l_{cl}(R)$ by Corollary 2.3.2. For instance, if $S \subseteq U(R)$, set of all invertible elements of R,(R is called a classical ring), then R is clearly an Ore ring with $Q^r_{cl}(R) = Q^l_{cl}(R) = R$. In particular any von Neumann regular ring is an Ore ring. Any commutative ring R is an Ore ring, by virtue of I.

IV Let R be a domain and $S = R\backslash\{0\}$. In this case, the right permutable condition (Definition 2.3.2) on S may be re-expressed in the equivalent form

$$aR \cap bR \neq (0) \qquad for\ a, b \in R\backslash\{0\} \tag{2.4.1}$$

is (right) Ore condition on R. Thus the domain R is right (resp. left) Ore if and only if R satisfies the right (resp. left) Ore condition.

Remark 2.4.1 Any division ring is an Ore domain.

Definition 2.4.1 (Right order) Let $R \subseteq Q$ be rings. We say that R is a right order in Q if

(i) every regular element of R is a unit in Q.

(ii) every element of Q has the form as^{-1}, where $a \in R$ and s is a regular element of R.

Left orders are defined similarly. If R is both a left and right order in Q, we shall simply say R is an order in Q.

Using this terminology, Lam [43] deduced the following result:

Proposition 2.4.1 The ring R is right ore if and only if it is a right order in some ring Q. In this case, $Q \cong Q^r_{cl}(R)$ over R. If moreover, R is a domain, then Q is a division ring and up to a unique R-isomorphism it is the only division hull of R.

Theorem 2.4.1 (Goldie [23]) For any domain R, the following are equivalent:

(i) R is right Ore domain.

(ii) u.dim$(R_R) = 1$

(iii) u.dim$(R_R) < \infty$
where u.dim(R_R) denotes the uniform dimension of R_R.

Proof $(i) \Leftrightarrow (ii) \Rightarrow (iii)$ are obvious . We finish by showing that $(iii) \Rightarrow (i)$. Assume that there exist $a, b \in R\backslash\{0\}$ such that $aR \cap bR = (0)$. Following A. Goldie, we show that $\{a^i b : i \geq 0\}$ are right R-linearly independent. Indeed if $\sum_{i \geq 0} a^i b r_i = 0$ where $r_i \in R$ are almost all zero, then
$br_0 + a(br_1 + abr_2 + ...) = 0 \Rightarrow r_0 = 0$ and
$br_1 + abr_2 + ... = 0$.
Repeating this argument, we see that all $r_i = 0$. Therefore, R contains $\oplus_{i>0} a^i bR$ (a free right module of countably infinite rank), so we have u.dim$(R_R) = \infty$.

Note that the equivalence of (ii) and (iii) is a special feature for domains and is false in general even for semisimple rings. For instance, if R is direct product of m division rings, then u.dim$(R_R) = m$, which can be any positive integer.

Corollary 2.4.1 If R is a right noetherian domain, then R is right Ore. In particular, $Q^r_{cl}(R)$ exists and it is the unique division hull of R.

Proof The noetherian module R_R cannot contain an infinite direct sum of nonzero submodules.

Corollary 2.4.2 If every finitely generated right ideal of a domain R is principal, then R is right Ore.

Proof Assume that $aR \cap bR = 0$ where $a, b \in R\backslash\{0\}$. Choose $c \in R$ such that $cR = aR \oplus bR$. Then $c = ar + bs$ and $b = cd$ for suitable $r, s, d \in R$. Right multiplying the former equation by d, we get $b = ard + bsd$, so $rd = 0$. This implies that $r = 0$, so $c = bs$ and hence $cR = bR$, contradicting $a \neq 0$.

Such a domain is known as Bezout domain.

Proposition 2.4.2 If a domain R is not an Ore domain, then R contains a copy of the free algebra $C < x_1, x_2, ... >$ where C is the center of R.

Proof Since R is not right Ore. Then there exist $a, b \in R_R$ that are right R-linearly independent. Jategaonkar's Lemma 2.2.3 then implies that the ring generated by a, b over C is isomorphic to $C < x, y >$. It follows from (Example 1.2, [41]) that the ring generated by $a, ab, ab^2, ...$ over C is isomorphic to $C < x_1, x_2, x_3... >$.

The following result is an immediate consequence of Proposition 2.4.2

Corollary 2.4.3 If a domain R is a PI-algebra over a field K, then R is an Ore domain.

Example 2.4.1 Let $R = \mathbb{Z}G$ where G is a group and $S = \mathbb{Z}\backslash\{0\}$. It is easy to see that $\mathbb{Q}G$, where $\mathbb{Q}$ is field of rational numbers is a right ring of fractions of R with respect to S. Therefore, the central localization RS^{-1} gives a ring naturally isomorphic to $\mathbb{Q}G$. Similarly, if R is the ring of quaternions $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{Z}$ and $S = \mathbb{Z}\backslash\{0\}$, then RS^{-1} is the ring of all rational quaternions. In this case, in fact, $RS^{-1} = Q^r_{cl}(R)$, since it is a division ring.

Example 2.4.2 Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. First choose $T = \{n.I \ : \ 0 \neq n \in \mathbb{Z}\}$. Using the method in Example 2.4.1, we see easily that the central localization RT^{-1} gives the ring $Q = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. It is easy to see that any regular element of R is a unit in Q. Therefore, R is an order in Q and $Q = Q^r_{cl}(R) = Q^l_{cl}(R)$. In particular, R is an Ore ring. Next, let us consider the multiplicative set

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ \mid \ a, b, c \in \mathbb{Z}, \ a \neq 0 \right\},$$

whose elements are not necessarily regular. Using the homomorphism $\varphi : R \to \mathbb{Q}$ defined by $\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a$, it is easy to check that $\mathbb{Q}$ is a right ring of fractions of R with respect to S. Therefore, RS^{-1} exists and is isomorphic to $\mathbb{Q}$. (The "Ore localization") here kills precisely all matrices of the form $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$). On the other hand $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is killed by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$ on the right, but for any $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq 0.$$

Therefore S is not left reversible, so $S^{-1}R$ does not exist.

Example 2.4.3 For a fixed prime p, let $\mathbb{Z}_p$ denote $\mathbb{Z}/p\mathbb{Z}$ and let $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. Proceeding as in the above example, let T be the central multiplicative set $\{n.I \ : \ n \in \mathbb{Z},\ p \nmid n\}$

(Of course, $n.I$ here means $\begin{pmatrix} n & 0 \\ 0 & \overline{n} \end{pmatrix}$.) The central localization RT^{-1} gives the ring

$$Q = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},$$

where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (p). (Note that $\mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$-module.) We can check easily that the multiplicative set of regular elements of R is

$$S = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \ \mid \ p \nmid x,\ z \neq 0 \in \mathbb{Z}_p \right\},$$

and that these elements are units in Q. Therefore, R is an order in Q and $Q = Q^r_{cl}(R) = Q^l_{cl}(R)$. In particular, R is an Ore ring. This fact can be checked directly as follows. To see that S is a right permutable, consider any $s = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in S$ and $a = \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \in R$. We can show that $aS \cap sR \neq \emptyset$ by solving the special matrix equation

$$\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \begin{pmatrix} u & 0 \\ n & w \end{pmatrix}.$$

This amounts to a single equation $ux = yu + zn$, which has a unique solution $n \in \mathbb{Z}_p$ since z is a unit in $\mathbb{Z}_p$. The fact that S is left permutable can be proved similarly. For later reference, let us note the following three additional properties of R :

(1) For $s \in R$, $\mathrm{l_R(s) = 0 \Rightarrow s \in S}$.

(2) The element $t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ has $\mathrm{r_R(t)} = 0$, but $t \notin S$.

(3) For t as above and $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have $aS \cap tR = \emptyset$.

To see (1), let $s = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \notin S$. If $p|x$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} s = 0$. If $p \nmid x$, we must have $z = 0$, in which case $\begin{pmatrix} 0 & 0 \\ -y & \overline{x} \end{pmatrix} s = 0$. For (2), note that $t \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} = \begin{pmatrix} pu & 0 \\ v & w \end{pmatrix}$ is zero only if $u = 0 \in \mathbb{Z}$ and $v = w = 0 \in \mathbb{Z}_p$. For (3), assume there is an equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ v & w \end{pmatrix}, \text{ with } \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in \mathrm{S}.$$

This leads to $x = pu$, a contradiction.

Example 2.4.4 Let $R = \begin{pmatrix} K & K[x] \\ 0 & K[x] \end{pmatrix}$, where K is a field. The multiplicative set of regular elements of R is

$$S = \left\{ \begin{pmatrix} c & f(x) \\ 0 & g(x) \end{pmatrix} \mid c \in K, \ \ f, g \in K[x], \ \ c.g \neq 0 \right\}.$$

It can be easily check that R is a right order in $Q = \begin{pmatrix} K & K(x) \\ 0 & K(x) \end{pmatrix}$. Therefore, $Q^r_{cl}(R) = RS^{-1} = Q$. On the other hand, $Q^l_{cl}(R)$ does not exists, as S turns out to be not left permutable. For $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ and $s = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in S$, a direct calculation shows that $Sa \cap Rs = \emptyset$. Therefore, the ring R is right Ore but not left Ore. Although every regular element of R becomes invertible in Q, the equation $Sa \cap Rs = \emptyset$ translate into the fact that $as^{-1} = \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix} \in Q$ cannot be written in the form of $t^{-1}r$ with $r \in R$ and t a regular element of R. therefore, Q is not a left ring of quotients of R with respect to S.

It is well known that twisted polynomial rings provide a rich source of examples of rings which exhibit different left and right behaviour.

Let σ be an endomorphism of a division ring R and $S = R[x;\sigma]$. Then S is a PLID in particular, it is left Ore by Corollary 2.4.1 and Corollary 2.4.2. If $\sigma(R) \neq R$ say $t \in R\backslash\sigma(R)$, then $[1, t] \subseteq R$ are right linearly independent over $\sigma(R)$. By Lemma 2.2.2 $[x, tx]$ are right linearly independent over S, so S is not right Ore (and hence not right noetherian). On the other hand, if $\sigma(R) = R$, then every left polynomial $\sum a_i x^i \in S$ is also a right polynomial and we can think of S as a ring of twisted right polynomials over R (with the twist rule $ax = x\sigma^{-1}(a)$ for $a \in R$). In this case S is a PRID and hence right noetherian and right Ore.

More generally, we can start with any domain R and try to find out when a twisted polynomial ring of the type $S = R[x;\sigma]$ is left Ore. We have the following result obtained by Lam [43].

Theorem 2.4.2 Let σ be an injective endomorphism of a domain R and let $S = R[x;\sigma]$. If R is left Ore, then so is S. The converse hold if σ is an automorphism.

Proof. We begin by noting that the injectivity of σ guarantees that S is also a domain. Assume S is left Ore and let $a, b \in R\backslash\{0\}$. Then $fa = gb$ for suitable $fg \in S\backslash\{0\}$. Considering the leading coefficients of both sides, we obtain an equation $c\sigma^n(a) = d\sigma^n(b)$ for some $c, d \in R\backslash\{0\}$ and $n \geq 0$. If σ is an automorphism we can apply σ^{-n} to get $Ra \cap Rb \neq 0$, so R is left Ore.

Conversely assume R is left Ore. Let K be the (unique) division ring of fractions of R. We can extend σ uniquely to an endomorphism of K by defining $\sigma as^{-1} = \sigma(a)\sigma s^{-1}$. Therefore, we can form $K[x;\sigma] \supseteq R[x;\sigma] = S$. Since $K[x;\sigma]$ is a PLID, it is left Ore by Corollary 2.4.1. Let Q be the division ring $Q^l_{cl}(K[x;\sigma])$. In view of proposition 2.4.1, it suffices to show that S is left order in Q. We have already $S\backslash\{0\} \subseteq U(Q)$. Next, each element of Q has the form $f^{-1}g$, where

$$0 \neq f = \sum a_i x^i, \quad g = \sum b_i x^i, \quad a_i, b_i \in K.$$

Choose a suitable common dennominator $s \in R\backslash\{0\}$ such that $a_i = s^{-1}c_i$, $b_i = s^{-1}d_i$ $(c_i, d_i \in R)$. Then

$$f^{-1}g = (s^{-1}\sum c_i x^i)^{-1}(s^{-1}\sum d_i x^i) = (\sum c_i x^i)^{-1}(\sum d_i x^i).$$

If δ is a derivation of the domain R, we can form the domain $R[x;\delta] = (\sum a_i x^i)$ using the law $xa = ax + \delta(a)$ for all $a \in R$. If R is left Ore, with division ring of fractions K, we can again extend δ (uniquely) to a derivation on K by defining;

$$\delta(as^{-1}) = \delta(a)s^{-1} - as^{-1}\delta(s)s^{-1} \quad (a \in R; 0 \neq s \in R)$$

The same proof used in Theorem 2.4.2 (with the couple of minor modification) yields the following analogue.

Theorem 2.4.3 Let δ be a derivation on the domain R. Then the differential polynomial domain $S = R[x; \delta]$ is a left Ore if and only if R is left Ore.

CHAPTER III

Maximal rings of quotients

3.1 Introduction

This chapter is devoted to the study of maximal rings of quotients based on the work of Utumi, Findlay and Lambek.

In 1956 Utumi [67] defined maximal ring of quotients of a ring as follows: Let Q be an over ring of a ring R. Then Q is said to be a right ring of quotients of R if R_R is a dense submodule of Q_R. The maximal right ring of quotients of R denoted by $Q^r_{max}(R)$ is the largest right ring of quotients of R. Analogously maximal left ring of quotients of R can be defined. There are examples of rings R with over rings Q such that Q strictly contains the classical ring of quotients of R, but still Q may be viewed as a kind of general ring of quotients of R. This leads us to the Findlay, Utumi, Lambek theory of maximal rings of quotients.

Utumi [67] showed that the maximal right ring of quotients always exists. In case R is a commutative domain with quotient field K, we have of course $Q^r_{cl}(R) = Q^r_{max}(R) = K$. Theorem 3.2.3 due to Lam [43] gives a sufficient condition for $Q^r_{max}(R)$ to be equal to $Q^r_{cl}(R)$ in case $Q^r_{cl}(R)$ exists.

Another point of view about maximal ring of quotients is given by Lambek [48]. He related the maximal ring of quotients theory with injective modules and pointed out that the maximal ring of quotients could be interpreted as the bicommutator of the injective envelope of R.

In section 3.3 some properties of semiprime rings are discussed which are crucial for the developement of the next section.

After discussing maximal rings of quotients, it is natural to include an introduction to the idea of Martindale rings of quotients of a ring. This kind of ring of quotients was introduced by Martindale [51] for prime rings in 1969 and by Amitsur [3] for semiprime rings in 1972. A more precise term for such rings of quotients is Martindale-Amitsur rings of quotients or two sided Martindale rings of quotients. Proposition 3.4.1 due to Lam characterize $Q_r(R)$, Martindale right ring of quotients. It completes section 3.4.

3.2 Alternate descriptions of maximal rings of quotients

The multiplicative set T is a two sided divisor set if it is both right and left divisor and in this case $RT^{-1} = T^{-1}R$. Lanning [48] proved that thare is a maximum two sided divisor set.

Theorem 3.2.1 Let R be a ring. If U and V are right (left) divisor sets in R. Then so is the multiplicatively closed set generated by them. Moreover, R has a maximum right (left) (two sided) divisor set.

Proof Let W be the multiplicatively closed set generated by right divisor sets U and V. Clearly every element of W is regular and $1 \in W$ so W is multiplicative set. Let $w \in W$ and $r \in R$. We will show that there exist $w' \in W$ and $r' \in R$ such that $wr' = rw'$. We prove this by induction on the minimum number n of nontrivial factors from U in the expression for w. If $n = 0$, then w is an element of the denominator set V so we are done. Thus assume that $n > 0$. Then we can write $w = xuv$, where $x \in W$, it can be expressed using $n-1$ factors from U and $u \in U$, $v \in V$. By the inductive hypothesis there exist $x' \in W$ and $r''' \in R$ such that $xr''' = rx'$. Because U is a right denominator set there exist $u' \in U$ such that $ur'' = r'''u'$ and then because V is a right denominator set there exist $v' \in V$ and $r' \in R$ such that $vr' = r''v'$. Setting $w' = x'u'v' \in W$ we have

$$wr' = xuvr' = xur''v' = xr'''u'v' = rx'u'v' = rw'$$

as required.

An analogous proof works for left denominator sets. Thus if T is the multiplicatively closed set generated by all right (left) (two-sided) denominator sets, then T is a right (left) (two-sided) denominator set and it is clear that it is the maximum one.

If T is the maximum two-sided divisor set of the ring R, then we define $Q_{cl}(R) = RT^{-1}$ to be the maximal symmetric Ore localization or maximal classical ring of quotients of R.

Utumi [67] defined maximal right (left) ring of quotients as follows:

Definition 3.2.1 (Maximal right ring of quotients) Let Q be an over ring of a ring R, then Q is said to be a right ring of quotients of R (general right ring of quotient) if R_R is a dense submodule of Q_R ($R_R \subseteq_d Q_R$). The maximal right ring of quotients of R is the largest right ring of quotients of R, written as $Q^r_{max} = Q^r_{max}(R)$.

Analogously one can define the maximal left ring of quotients of R, written as $Q^l_{max} = Q^l_{max}(R)$.

The following example shows that $Q^r_{cl}(R)$ is a general right ring of quotients:

Example 3.2.1 Let R be a ring such that $Q^r_{cl}(R)$ exists. Let $x, y \in Q^r_{cl}(R)$, then we can write $x = as^{-1}$ and $y = bs^{-1}$ for $a, b \in R$ and s a regular element of R. Then $ys = b \in R$ and $xs = a \neq 0$ if $x \neq 0$. Hence $Q^r_{cl}(R)$ is a right ring of quotients of R.

Contrary to the situation of the right classical ring of quotients $Q^r_{cl}(R)$, Utumi [67] proved that the maximal right ring of quotients always exists.

The following result due to Lam [43] reflects the maximality of $Q^r_{max}(R)$.

Theorem 3.2.2 Let T be any general right ring of quotients of a ring R and let $Q = Q^r_{max}(R)$. Then

(i) There exists a unique ring homomorphism $g : T \to Q$ extending the identity map on R.

(ii) The homomorphism g above is one-one.

(iii) The ring structure on Q is the only one extending the R-module structure on Q_R.

Proposition 3.2.1 For right R-modules $N \subseteq M$, the following conditions are equivalent :

(i) $N \subseteq_d M$.

(ii) $Hom_R(M/N, E(M)) = 0$, where $E(M)$ is the injective hull of M.

(iii) For any submodule P such that $N \subseteq P \subseteq M$, $Hom_R(P/N, M) = 0$.

Proof (i) $\Rightarrow$ (ii). Assume that there exists a nonzero R-homomorphism $f : M \to E(M)$ with $f(N) = 0$. Then $M \cap f(M) \neq 0$ so there exists $x, y \in M\backslash\{0\}$ such that $f(y) = x$. By (i) there exists $r \in R$ with $xr \neq 0$ and $yr \in N$. But then $0 = f(yr) = f(y)r = xr$, a contradiction.

(ii) $\Rightarrow$ (iii). Suppose that, for some P as in (iii), there exists a nonzero R-homomorphism $g : P/N \to M$. By the injectivity of $E(M)$, we can extend g to a (nonzero) $M/N \to E(M)$.

(iii) $\Rightarrow$ (i). Suppose that $x.y^{-1}N = 0$ for some $y \in M$, $x \in M\backslash\{0\}$. We define $f : N + yR \to M$ by

$$f(n + yr) = xr \quad (n \in N,\ r \in R).$$

This map is well defined, for, if $n + yr = n' + yr'$ then $n - n' = y(r' - r) \in N$, hence $x(r - r') = 0$. Clearly f is an R-homomorphism vanishing on N, so by (iii), $0 = f(y) = x$, a contradiction.

Proposition 3.2.2 Let M be a right R-module such that $M \subseteq_d P$. Then there exists a unique R-homomorphism $g : P \to \widetilde{E}(M)$ where $\widetilde{E}(M) = \{i \in I \mid \forall\ h \in H,\ h(M) = 0 \Rightarrow h(i) = 0\}$ extending the inclusion map $M \hookrightarrow \widetilde{E}(M)$. This g is necessarily one-to-one.

Proof Since $M \subseteq_e P$, the inclusion $M \to E(M)$ extends to an embedding $g : P \to E(M)$. Clearly $M \subseteq_d g(P)$ so by Theorem 1.3.1 $g(P) \subseteq \widetilde{E}(M)$. Now suppose $g_1, g_2 : P \to \widetilde{E}(M)$ both extend the inclusion map $M \to \widetilde{E}(M)$. Since $M \subseteq_e P$, the $g_i's$ are monomorphism. Consider the map $f : g_1(P) \to \widetilde{E}(M)$ defined by $f(g_1(p)) = g_1(p) - g_2(p)$ $(p \in P)$. Since $f(M) = 0$ and $M \subseteq_d \widetilde{E}(M)$, we must have $f = 0$ by Proposition 3.2.1, so $g_1(p) = g_2(p)$ for all $p \in P$.

Proposition 3.2.3 Let M be an R-module containing the right regular module R_R. Then $R_R \subseteq_d M$ if and only if $R_R \subseteq_e M$ and for every $y \in M$, $y^{-1}R \subseteq_d R_R$.

Proof of the Theorem 3.2.2 By Proposition 3.2.2, there exist a unique R-homomorphism $g : T \to \widetilde{E}(R) = Q$ extending the identity map on R and g is one-one. If we can show that $g(t't) = g(t')g(t)$ for all $t, t' \in T$, then clearly all three parts of the Theorem follow. Let $h \in H$ extend the R-homomorphism $\widetilde{E}(R) \to \widetilde{E}(R)$ given by left multiplication by $g(t't) - g(t')g(t)$. For every $r \in t^{-1}R := \{x \in R : tx \in R\}$ we have

$$\begin{aligned} h(r) &= (g(t't) - g(t')g(t))r \\ &= g(t'tr) - g(t')g(tr) \\ &= g(t')(tr) - g(t')(tr) = 0 \end{aligned}$$

Therefore, $h(t^{-1}R) = 0$. But $R_R \subseteq_d T_R$ implies that $t^{-1}R \subseteq_d R_R$ (cf by Proposition 3.2.3) and hence by Proposition 3.2.1 $h(R) = 0$. In particular, $0 = h(1) = g(t't) - g(t')g(t)$.

Corollary 3.2.1 (Lam [43]) If $Q^r_{cl}(R)$ exists, then it has a unique embedding in the ring $Q^r_{max}(R)$ extending the identity map on R.

In case when R is a commutative domain with quotient field K, we have, of course $Q^r_{cl}(R) = Q^r_{max}(R) = K$. The following example shows that in general, $Q^r_{max}(R)$ may be bigger than $Q^r_{cl}(R)$, if the latter exists.

Example 3.2.2 Let R be the ring of upper traingular $n \times n$ matrices over a semisimple ring k. Since R is artinian, all regular elements are units, so $R = Q^r_{cl}(R)$. On the other hand, $E(R_R) = \mathbb{M}_n(k)$. We claim that $R_R \subseteq_d \mathbb{M}_n(k)$. Once we have shown this, it will follow that $Q^r_{max}(R) = \mathbb{M}_n(k)$ as rings, by Theorem 3.2.2(iii). To show the denseness of R, let $x = (x_{ij})$, $y = (y_{ij})$ be $n \times n$ matrices, where $x \neq 0$. Choose $s \in R$ with last column $(a_1, ..., a_n)^t$ and all other columns zero. Clearly $ys \in R$ and, choosing $(a_1, ..., a_n)$ to be the j^{th} unit vector where $x_{ij} \neq 0$ for some i, we also have $xs \neq 0$. This proves the claim that $R_R \subseteq_d \mathbb{M}_n(k)$. Similarly, it can be shown that ${}_RR \subseteq_d \mathbb{M}_n(k)$. Since $\mathbb{M}_n(k)$ is also the injective hull of ${}_RR$ we deduce as before that $Q^l_{max}(R) = \mathbb{M}_n(k)$, while $Q^l_{cl}(R) = R$.

The following result gives a sufficient condition for $Q^r_{max}(R)$ to be equal to

$Q^r_{cl}(R)$, in case $Q^r_{cl}(R)$ exists.

Theorem 3.2.3 Suppose $Q^r_{cl}(R)$ exists and every dense right ideal of the ring R contains a regular element. Then $Q^r_{max}(R) = Q^r_{cl}(R)$.

Proof We shall show that every $q \in Q = Q^r_{max}(R)$ belongs to $Q^r_{cl}(R)$. Since $R_R \subseteq_d Q_R$, so it follows that $q^{-1}R \subseteq_d R_R$ by Proposition 3.2.3. By hypothesis, $q^{-1}R$ contains a regular element s of R. Then $a := qs \in R$. Since $s \in U(Q^r_{cl}(R))$, we have $q = as^{-1} \in Q^r_{cl}(R)$, as desired.

Corollary 3.2.2 If R is a semiprime right Goldie ring, then $Q^r_{max}(R) = Q^r_{cl}(R)$ and this is a semisimple ring.

Corollary 3.2.3 (Small [64]) If R is a commutative ring with ACC on annihilator ideals, then $Q^r_{max}(R) = Q^r_{cl}(R)$.

Proof By (Theorem(8.31)(1), [43]), every dense ideal of R contains a regular element, so Theorem 3.2.3 applies.

Recall that if a ring R is Ore, so that $Q^r_{cl}(R)$ and $Q^l_{cl}(R)$ both exist, then they are isomorphic over R. If R is also a domain, then Corollary 3.2.2 and (Corollary 11.20, [43]) implies that $Q^r_{cl}(R) = Q^r_{max}(R)$ and $Q^l_{cl}(R) = Q^l_{max}(R)$. Moreover $Q^r_{max}(R) \cong Q^l_{max}(R)$ over R.

Another point of view about the maximal ring of quotients was given by Lambek [46]. He related the maximal ring of quotients theory with injective modules and pointed out that the maximal ring of quotients could be interpreted as the bicommutator of the injective envelope of R, as follows.

Let $I := E(R)$, the injective hull of the right regular module R_R. Let $H = End(I_R)$ operating on the left of an ideal I of R. Furthermore, let $Q = End({}_HI)$, operating on the right of I. So we have that $I =_H I_Q$. The ring Q is referred to as the double commutant of R and by [46] we have that $Q^r_{max}(R) = Q$.

This construction leads the following result:

Theorem 3.2.4 (Passman [55]) If R is right self injective, then $Q^r_{max}(R) = R$

In [43] Lam has given another description for $Q^r_{max}(R)$ for any ring R in terms of dense right ideals of R. For this we need the following criterion for dense R-submodules of $Q = Q^r_{max}(R)$.

Proposition 3.2.4 An R submodule $D_R \subseteq Q_R$ is dense in Q_R if and only if for any $h \in H$, $h(D) = 0$ implies that $h(1) = 0$.

Proof Assume first $D \subseteq_d Q$ and suppose $h \in H$ is such that $h(D) = 0$. Then

$$h : Q \to I = E(R) = E(Q)$$

must be zero by Proposition 3.2.1. In particular, $h(1) = 0$. Conversely, suppose $h(D) = 0 \Rightarrow h(1) = 0$, for any $h \in H$. Let P be any right R-module between D and Q. Any R-homomorphism $f : P \to Q$ is the restriction of some $h \in H$, by the injectivity of I_R. Thus, if $f(D) = 0$, we have $h(D) = 0$ and so $h(1) = 0$. Therefore,

$$f(P) = h(1.P) = h(1)P = 0.$$

This shows that $Hom_R(P/D, Q) = (0)$, so by Proposition 3.2.1, $D \subseteq_d Q$, as desired.

Proposition 3.2.5 Let D, D' be R-submodules of Q_R such that $D \subseteq_d Q$. Then $Hom_R(D, D')$ is isomorphic (as a group) to

$$E = E_{D.D'} := \{q \in Q : qD \subseteq D'\}.$$

In particular, $End(D_R)$ is isomorphic to the subring $E_{D.D} \subseteq Q$ and each R-homomorphism from D to R_R is given by left multiplication by a suitable $q \in Q$.

Proof Define $\psi : E \to Hom_R(D, D')$ by $\psi(q)(d) = qd$ where $q \in E$ and $d \in D$. Clearly, $\psi(q)$ is a (right) R-homomorphism. If $\psi(q) = 0$, then $qD = 0$. Write $q = h.1$ where $h \in H$. Then

$$h(D) = h(1.D) = (h.1)D = qD = 0.$$

Since $D \subseteq_d Q$, Proposition 3.2.4 implies that $q = h.1 = 0$. To show that ψ is onto, consider any $f \in Hom_R(D, D')$. We may assume that f is the restriction of some $h \in H$. Now let $q = h.1 \in I$. We claim that $q \in Q$. Indeed, for every $h' \in H$ such that $h'(R) = 0$ we have

$$(h'h)(D) = h'(hD) = h'(f(D)) \subseteq h'D' \subseteq h'Q = 0.$$

This implies $0 = (h'h)(1) = h'(q)$. Therefore, we have $q \in Q$ and now

$$f(d) = h(d) = (h.1)d = qd = \psi(q)(d)$$

for all $d \in D$, so $f = \psi(q)$. In the case when $D' = D$, the map ψ is clearly a ring homomorphism, this gives the last conclusion in the Proposition.

Note that, in the above if we let R be a commutative ring for which $Q^r_{max}(R) = Q^r_{cl}(R)$, Then Proposition 3.2.5 would give back ((2.16'), [43]). However Proposition 3.2.5 applies more generally to any ring R. With its help, we arrive at the following alternate description of $Q^r_{max}(R)$.

Theorem 3.2.5 $Q^r_{max}(R)$ can be identified as a ring whose elements are classes of R-homomorphisms $A \to R$ where A is any dense right ideal of R. Two such R-homomorphisms $f : A \to R$, $f' : A' \to R$ are regarded to be in the same class if $f = f'$ on $A \cap A'$. The classes of $f : A \to R$ and $g : B \to R$ are added by taking the class of $f + g : A \cap B \to R$ and they are multiplied by taking the class of $fg : g^{-1}(A) \to R$.

Proof To see that the description of multiplication is meaningful, note that since $A \subseteq_d R_R$, the preimage $g^{-1}(A)$ is dense in B by Theorem 1.3.5 and hence dense in R_R by Theorem 1.3.6(ii). Therefore $fg : g^{-1}(A) \to R$ does define a class. For $q \in Q$, Theorem 1.3.5 also implies that $q^{-1}R \subseteq_d R_R$, so left multiplication by q gives a right R-homomorphism $q^{-1}R \to R$.

Conversely, for $A \subseteq_d R_R$, any R homomorphism $f : A \to R$ is given by left multiplication by a unique $q \in Q$, according to Proposition 3.2.5. For such an element q, we have $qA \subseteq R$ so $A \subseteq q^{-1}R$. The remaining details are easy to proof.

Another important application of Proposition 3.2.5 is given by the following result of Utumi [68]

Theorem 3.2.6 Suppose R has a minimal dense right ideal D (i.e. R is right artinian). Then

(*i*) $D \subseteq D'$ for any right ideal $D' \subseteq_d R_R$.

(*ii*) D is an ideal of R containing the right socle $soc(R_R)$.

(*iii*) $Q^r_{max}(R) \cong End(D_R)$ as rings.

Proof (i) follows from the result that $D \cap D' \subseteq_d R_R$ (By Theorem 1.3.6(i)). Consider any $q \in Q$. Since $D \subseteq_d Q_R$, Theorem 1.3.5 gives

$$q^{-1}D = \{r \in R : qr \in D\} \subseteq_d R_R.$$

By (i), we have $D \subseteq q^{-1}D$, so $qD \subseteq D$. In particular, $R.D \subseteq D$ so D is an ideal of R. Since $D \subseteq_e (R_R)$, any minimal right ideal of R is contained in D; hence $soc(R_R) \subseteq D$. Finally Proposition 3.2.5 gives (iii).

Corollary 3.2.4 Let R be a right Kasch ring. Then $Q^r_{max}(R) = R$. (In particular, if R is a commutative Kasch ring, then $Q_{max}(R) = Q_{cl}(R) = R$.)

Proof By (Corollary 8.28, [43]), the only dense right ideal in R is R itself. Thus, we can apply Theorem 3.2.6 to $D = R$. For this choice of D, $Q^r_{max}(R) \cong End(D_R) \cong R$. Of course this conclusion can be obtained directly: for $q \in Q^r_{max}(R)$ we have $q^{-1}R \subseteq_d R_R$, so $q^{-1}R = R$. Therefore $1 \in q^{-1}R$, whence $q = q.1 \in R$. The last statement of the Corollary now follows from Corollary 3.2.1.

Corollary 3.2.5 Let R be a right nonsingular right artinian ring. Then $S := soc(R_R)$ is the smallest dense right ideal of R and $Q^r_{max}(R) \cong End(S_R)$.

The two results above enable us to give a few more explicit computations of $Q = Q^r_{max}(R)$.

Example 3.2.3 Let (R, m) be a local ring, where $m = rad\ R$ is nilpotent. Then R is a (right) Kasch ring, so by Corollary 3.2.4, $Q^r_{max}(R) = R$. If we take R to be the local 3-dimensional k-algebra, where k is a field, then we have

$$R = Q^r_{cl}(R) = Q^r_{max}(R) \subset E(R),$$

since $dim_k E(R) = 6$.

Example 3.2.4 Let $R = \begin{pmatrix} A & 2A \\ 0 & A \end{pmatrix}$, where $A = \mathbb{Z}_4$. This ring of 32 elements is not (left, right) nonsingular, but it is Kasch. Thus, we conclude from Corollary 3.2.4 that $R = Q^r_{max}(R) \subset E(R_R)$. This implies that no ring properly containing R can be a general right ring of quotients of R. Take the ring $T = \begin{pmatrix} A & 2A \\ A & A \end{pmatrix} \supset R$. For

$x = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in T,

it is clear that $yr \in R$ implies $a = 0$ and hence necessarily $xr = 0$. This shows that R_R is not dense in T_R, so T is indeed not a general right ring of quotients of R. The same is true for the smaller ring $T' = \begin{pmatrix} A & 2A \\ 2A & A \end{pmatrix} \supset R$. We choose $y = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ in this case and note that $yr \in R \Longrightarrow a \in 2A \Longrightarrow xr = 0$.

Example 3.2.5 Let R be the (artinian) ring of upper triangular $n \times n$ matrices over a semisimple ring k. This ring is right nonsingular so Corollary 3.2.5 applies. In (Example 7.14b, [43]) we have observed that $S := soc(R_R)$ is the ideal of all matrices with zeros on all but the last column. Identifying such matrices with their last columns, we may view S as k^n. Here, any $a = (a_{ij}) \in R$ acts on the right of a column vector by right multiplication by a_{nn}. Therefore,

$$Q^r_{max}(R) \cong End(S_R) \cong End(k^n)_k \cong \mathbb{M}_n(k).$$

Under the composite isomorphism, elements of R do correspond to themselves as upper triangular matrices. Of course, this computation of $Q^r_{max}(R)$ is in agreement with the earlier one given in Example 3.2.2. A similar computation with $soc(_RR)$ shows that $Q^l_{max}(R) = \mathbb{M}_n(k)$, again with elements of R corresponding to themselves.

Example 3.2.6 Let $R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is as in Example 3.2.5. Then R is artinian and nonsingular, with

$$S = soc(R_R) = \begin{pmatrix} 0 & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}, \qquad soc(_RR) = \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the latter, we get $Q^l_{max}(R) \cong \mathbb{M}_3(k)$. From the former, we get a decomposition $S_R = A \oplus B$ where

$$A = \begin{pmatrix} 0 & k & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}.$$

After computing the actions of R on A and B, we see that A, B are the isotypic components of the semisimple module S_R and deduce that

$$\begin{aligned} Q^r_{max}(R) &\cong End(A \oplus B)_R \\ &\cong EndA_R \times EndB_R \\ &\cong EndA_k \times EndB_k \\ &\cong \mathbb{M}_2(k) \times \mathbb{M}_2(k) \end{aligned}$$

Thus as long as $k \neq 0$, the two rings $Q^r_{max}(R)$ and $Q^l_{max}(R)$ are not isomorphic.

Example 3.2.7 Let k be a nonzero semisimple ring and $V = k \oplus ... \oplus k$ (n copies), viewed in the natural way as a (k, k)-bimodule. Let R be the triangular ring $\begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$, which is, of course, artinian. we have here

$$S = soc(R_R) = \begin{pmatrix} 0 & V \\ 0 & k \end{pmatrix} \quad \text{and} \quad S' = soc(_RR) = \begin{pmatrix} k & V \\ 0 & 0 \end{pmatrix}.$$

So by (Proposition 7.13, [43]), R is nonsingular. Using Corollary 3.2.5, it follows as in the Example 3.2.6 that

$$Q^r_{max}(R) \cong End(S_R) \cong End(k^{n+1})_k \cong \mathbb{M}_{n+1}(k), \quad \text{and}$$

$$Q^l_{max}(R) \cong (R) \cong End(_RS') \cong End_k(k^{n+1}) \cong \mathbb{M}_{n+1}(k).$$

While these two maximal rings of quotients are isomorphic as rings, they are not isomorphic over R if $n \geq 2$. In fact, if $\varphi : Q^r_{max}(R) \to \mathbb{M}_{n+1}(k)$ and $\psi : Q^l_{max}(R) \to \mathbb{M}_{n+1}(k)$ are the isomorphisms given above, we can check easily that, for $a, b \in k$ and $v = (v_1, ..., v_n) \in k^n$:

$$\varphi\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} & & & v_1 \\ & aI_n & . & \\ & & & v_n \\ 0 & ..... & 0 & b \end{pmatrix} \text{ and } \psi\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & v_1 & .... & v_n \\ 0 & & & \\ & & bI_n & \\ 0 & & & \end{pmatrix}.$$

Of course, these matrix representations depend on a specific labeling of the usual basis of k^{n+1}. Let $A = \varphi(R) \cong R$ and $B = \psi(R) \cong R$. We know that

$$E(A_A) = \mathbb{M}_{n+1}(k) \quad \text{and} \quad E({}_BB) = \mathbb{M}_{n+1}(k).$$

Moreover, it is easy to see that $A \subseteq_d \mathbb{M}_{n+1}(k)$ as right A-modules. Infact, if $x, y \in \mathbb{M}_{n+1}(k)$ with $x \neq 0$, there exists $\begin{pmatrix} v \\ b \end{pmatrix} \in k^{n+1}$ with $x.\begin{pmatrix} v \\ b \end{pmatrix} \neq 0$. For $\alpha := \begin{pmatrix} 0 & v \\ 0 & b \end{pmatrix} \in A$, we have then $x.\alpha \neq 0$ and $y.\alpha \in A$. It follows that $Q^r_{max}(A) = \mathbb{M}_{n+1}(k)$ and similarly $Q^l_{max}(B) = \mathbb{M}_{n+1}(k)$. This provides alternative computations to $Q^r_{max}(R)$ and $Q^l_{max}(R)$. On the other hand, it is easy to check that B is not essential in $\mathbb{M}_{n+1}(k)_B$, so $\mathbb{M}_{n+1}(k) \supseteq B$ is not a maximal right ring of quotients of B. Similarly, $\mathbb{M}_{n+1}(k) \supseteq A$ is not a maximal left ring of quotients of A. There is no contradiction here; this simply means that $Q^r_{max}(R)$ and $Q^l_{max}(R)$ cannot be isomorphic over R.

In 1991 Passman [55] proved the following Theorem which characterize Q^r_{max}. For any ring R, let $\mathcal{D} = \mathcal{D}(R)$ be the collection of all dense right ideal of R:

Theorem 3.2.7 The maximal right ring of quotients $Q^r_{max}(R)$ of the ring R satisfies

(i) R is a subring of $Q^r_{max}(R)$.

(ii) For all $q \in Q^r_{max}(R)$ there exist $J \in \mathcal{D}(R)$ such that $qJ \subseteq R$, where $\mathcal{D}(R)$ is the collection of all dense right ideal of R.

(*iii*) For all $q \in Q^r_{max}(R)$ and $J \in \mathcal{D}(R)$, $qJ = 0$ iff $q = 0$. (iv) For all $J \in \mathcal{D}(R)$ and $f : J_R \to R_R$ there exists $q \in Q^r_{max}(R)$ such that $f(x) = qx$ for all $x \in J$. Furthermore properties (i) – (iv) characterize ring $Q^r_{max}(R)$ up to isomorphism.

Proof We have only to prove last statement . Let $Q \supseteq R$ be a ring having properties (i) - (iv). Define the mapping $\alpha : Q \to Q^r_{max}(R)$ by the rule $q^\alpha = [l_q; (q : R)_R]$, where l_q is left multiplication determined by q. One can easily check that α is an isomorphism of rings identical on R.

Lemma 3.2.1 (Utumi [67]) let R be any ring. Then the following hold

(*i*) If T is right quotient ring of R, then $Q^r_{max}(R) = Q^r_{max}(T)$.

(*ii*) $Q^r_{max}(Q^r_{max}(R)) = Q^r_{max}(R)$.

By the above result, it is clear that $Q^r_{max}(R)$ is a closure operation on rings. Using the characterization of $Q^r_{max}(R)$ of Theorem 3.2.7, it can be shown that the maximal ring of quotients of a direct product of rings is the direct product of the corresponding maximal rings of quotients.

$$Q^r_{max}(\Pi R_i) \cong \Pi Q^r_{max}(R_i)$$

because the right dense ideals of ΠR_i are exactly the right ideals of the form ΠD_i where D_i are right dense ideals of R_i for all i.

Definition 3.2.2 (Center of $Q^r_{max}(R)$) Let R be any ring. Then the centre of $Q^r_{max}(R)$ is the set of elements of $Q^r_{max}(R)$ that commute with R and is called the extended centroid of R.

Theorem 3.2.8 Let R be any ring and Q be a right ring of quotients of R. Then $\mathcal{Z}(R_R) = \mathcal{Z}(Q_Q) \cap R$. In particular R is right nonsingular if and only if Q is right nonsingular.

Proof First we see that $\mathcal{Z}(R_R) \subseteq \mathcal{Z}(Q_Q)$. This comes from the fact that if Q is a right quotient ring of R and I is right essential ideal of R, then IQ is a right essential ideal of Q.

Conversely, it is clear that $\mathcal{Z}(Q_Q) \cap R$ is contained in $\mathcal{Z}(R_R)$ since for every right essential ideal I of Q we have that $I \cap R$ is a right essential ideal of R

3.3 Dense ideals in semiprime rings

For any submodule J of a right R-module M and any subset $S \subseteq M$, set

$$(S : J)_R = \{x \in R \mid Sx \subseteq J\}$$

Proposition 3.3.1 (Beidar [5]) Let R be a semiprime ring and I, J, $S \in \mathcal{D}(R)$, where $\mathcal{D} = \mathcal{D}(R)$ is the collection of all dense right ideal of R and let $f : I \to R$ be a homomorphism of right R modules Then

(i) $f^{-1}(J) = \{a \in I / f(a) \in J\} \in \mathcal{D}(R)$

(ii) $(a : J) \in \mathcal{D}(R)$ for all $a \in R$

(iii) $I \cap J \in \mathcal{D}(R)$

(iv) If K is a right ideal of R and $K \supseteq I$, then $K \in \mathcal{D}(R)$

(v) $l(I) = 0 = r(I)$

(vi) If K is a right ideal of R and $(a : K) \in \mathcal{D}(R)$ for all $a \in I$, then $K \in \mathcal{D}(R)$

(vii) If L is a right ideal of R and $g : L \to R$ is a homomorphism of right S-modules, then g is a homomorphism of right R-modules

($viii$) $IJ \in \mathcal{D}(R)$

Proof (i) Let $r_1 \neq 0$, $r_2 \in R$ Since I is a dense right ideal of R, $r_1 r' \neq 0$ and $r_2 r' \in I$ for some $r' \in R$ Analogously $(r_1 r')r'' \neq 0$ and $f(r_2 r')r'' \in J$ for some $r'' \in R$ Setting $r = r'r''$ we conclude that $r_1 r \neq 0$ and $r_2 r \in f^{-1}(J)$, which means that $f^{-1}(J)$ is a dense right ideal of R

(ii) Letting l_a denote the left multiplication by a Let $x \in (a : J)$ Then $ax \in J$, this implies that $l_a(x) \in J$ i e $x \in l_a^{-1}(J) \Rightarrow (a : J) \subseteq l_a^{-1}(J)$

Conversely let $r \in l_a^{-1}(J)$. Then $l_a(r) \in J$ and $ar \in J$ i.e. $r \in (a : J)$. Thus $l_a^{-1}(J) \subseteq (a : J)$. This implies that $(a : J) = l_a^{-1}(J)$ and now apply (i). Hence $(a : J) \in \mathcal{D}(R)$.

(iii) If I is the inclusion map $I \to R$, then $I \cap J = i^{-1}J$. Now apply (i). We get $I \cap J \in \mathcal{D}(R)$.

(iv) Since I is dense right ideal. For any $r_1 \neq 0 \in R$ and $r_2 \in R$ $\exists$ $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in I$, as $I \subseteq K$
$\Rightarrow r_2 r \in K$
$\Rightarrow K$ is a dense right ideal.

(v) Suppose $Ia = 0$ for some $0 \neq a \in R$. Setting $r_1 = a = r_2$ we see that there exists $r \in R$ such that $0 \neq ar \in I$. We then have a contradiction $arRar \subseteq Iar = 0$. Next, we suppose $l(I) \neq 0$. Since R is semiprime , there exists $a, b \in l(I)$ such that $ab \neq 0$. Now we can find $r \in R$ such that $abr \neq 0$ and $br \in I$. But $abr \in aI = 0$ again a contradiction is reached.

(vi) Let $0 \neq r_1, r_2 \in R$. Since $I \in \mathcal{D}(R)$, there is an element $r' \in R$ such that $r_1 r' \neq 0$ and $r_2 r' \in I$. Hence $(r_2 r' : K) \in \mathcal{D}(R)$. By part (v) we then have $l((r_2 r' : K)) = 0$ and hence $r_1 r' r'' \neq 0$ and $r_2 r' r'' \in K$ for some $r'' \in (r_2 r' : K)$. Thus $K \in \mathcal{D}(R)$.

(vii) Let $x \in L$ and $r \in R$. By (ii) $(r : S)_R \in \mathcal{D}(R)$ and so by (iii) $M = (r : S)_S \cap S \in \mathcal{D}(R)$. For every $y \in M \subseteq S$ we have $ry \in S$ and so (g(xr)-g(x)r)y = g(xr)y-g(x)(ry) = g(xry)-g(xry) = 0.
It follows from (v) that $g(xr) = g(x)r$ and thus g is a homomorphism of right R-modules.

(viii) Let $r_1 \neq 0$ and $r_2 \in R$. By (ii) $L = (r_2 : I) \in \mathcal{D}(R)$, and so by (v) there exists $r' \in L$ such that $r_1 r' \neq 0$ and r" $\in J$ such that $r_1 r' r'' \neq 0$. Setting $r = r' r''$ we then have $r_1 r \neq 0$ and $r_2 r = (r_2 r') r'' \in IJ$.

Corollary 3.3.1 Let R be a semiprime ring and J be a right ideal of R. Then $J \in \mathcal{D}(R)$ if and only if $l_R((a : J)) = 0$ for all $a \in R$.

Proof If $J \in \mathcal{D}$ we know from proposition 3.3.1(ii) and (v) that $(a : J) \in \mathcal{D}$ and $l_R((a : J)) = 0$. Conversely, given $r_1 \neq 0$, $r_2 \in R$ we know that $r_1(r_2 : J) \neq 0$, and so we may choose $r \in (r_2 : J)$ such that $r_1 r \neq 0$. But since $r \in (r_2 : J)$, we also have $r_2 r \in J$

Remark 3.3.1 Let R be a semiprime ring and J be a right ideal of R. If $f : J \to R$ is a right R-module homomorphism, then

(*i*) If $a \in R$ and $r(a) \in \mathcal{D}(R)$, then $a = 0$.

(*ii*) If $Kerf \in \mathcal{D}(R)$, then $f = 0$.

Proof (i) The first statement follows from Proposition 3.3.1(v).

(ii)Suppose that $Ker(f) \in \mathcal{D}(R)$. Then we have $f(b)(b : Ker(f)) = 0$ for all $b \in J$. According to Proposition 3.3.1(ii), $(b : Ker(f)) \in \mathcal{D}(R)$. By the first statement we then have $f(b) = 0$. Thus $f = 0$

Lemma 3.3.1 Let R be a semiprime ring and $q_1, q_2..., q_n \in Q^r_{max}(R)$ and $I, J \in \mathcal{D}(R)$. Then there exists $L \in \mathcal{D}(R)$ such that $L \subseteq J$ and $q_i L \subseteq I$ for all $i = 1, 2, ...n$.

Proof Setting $J_i = (q_i : R)_R$ noting that $J_i \in \mathcal{D}$ for all i. Consider the map $f_i = l_{q_i} : J_i \to R_R$. By Proposition 3.3.1 $K_i = f_i^{-1}(I) = \{x \in J_i \mid q_i x \in I\} \in \mathcal{D}$. Setting $L = (\cap_{i=1}^n K_i) \cap J$, we have the desired dense right ideal.

Lemma 3.3.2 Let R be a semiprime ring and K be a dense right ideal of a semiprime ring R and S a subring of $Q^r_{max}(R)$ such that $K \subseteq S$. Then

(*i*) S is a semiprime ring;

(*ii*) A right ideal J of S is dense if and only if $(J \cap R)K \in \mathcal{D}(R)$ (in particular $IS \in \mathcal{D}(S)$ if $I \in \mathcal{D}(R)$).

(iii) A right ideal J of S is essential if and only if $(J \cap R)K$ is an essential right ideal of R.

Proof Assume that I is a nonzero nilpotent ideal of S and pick $0 \neq q \in I$. By Theorem 3.2.7(ii) $qJ \subseteq R$ for $J \in \mathcal{D}(R)$ and by Theorem 3.2.7(iii) $0 \neq q(J \cap K) \subseteq I \cap R$ is a nonzero nilpotent right ideal of the semiprime ring R, a contradiction.

(ii) Suppose that $J \in \mathcal{D}(S)$ and $r_1 \neq 0$, $r_2 \in R$. Since $K \in \mathcal{D}(R)$, $L = (r_1 : K)_R \cap (r_2 : K)_R \in \mathcal{D}(R)$. By Proposition 3.3.1(v) $r_1 r' \neq 0$ for some $r' \in L$. Clearly $r_1 r', r_2 r' \in K \subseteq S$. Therefore there exists an element $q \in S$ such that $r_1 r' \neq 0$ and $r_2 r' q \in J$. Again by Proposition 3.3.1 we have that $r_1 r' q r'' \neq 0$ for some $r'' \in (q : R)_R \cap K$. Clearly $r_2 r' q r'' \in J \cap R$. Pick $r''' \in K$ such that $r_1 r' q r'' r''' \neq 0$. Since $r_2 r' q r'' r''' \in (J \cap R)K$ and $r' q r'' r''' \in R$, we conclude that $(J \cap R)K \in \mathcal{D}(R)$.

Conversely, let $(J \cap R)K \in \mathcal{D}(R)$ and $s_1 \neq 0$, $s_2 \in S$. According to Proposition 3.3.1 we have $s_1 r' \neq 0$ for some $r' \in (s_1 : R)_R \cap (s_2 : R)_R \cap K$. Clearly $s_1 r', s_2 r' \in R$ and $r' \in K \subseteq S$. Since $(J \cap R)K \in \mathcal{D}(R)$. $(s_2 r' : (J \cap R)K)_R \in \mathcal{D}(R)$ and therefore $L = (s_2 r' : (J \cap R)K)_R \cap K \in \mathcal{D}(R)$. Hence there exists an element $r'' \in L$ such that $s_1 r' r'' \neq 0$. Note that $s_2 r' r'' \in (J \cap R)K \subseteq J$ and $r', r'', r' r'' \in K \subseteq S$. Therefore J is a dense right ideal of S. In particular, if $I \in \mathcal{D}(R)$, then $(IS \cap R)K \supseteq (IK \cap R)K \supseteq IK^2$ and so $IS \in \mathcal{D}(S)$.

(iii) Assume that J is an essential right ideal of S. We set $I = J \cap R$. Let M be any nonzero right ideal of R. By Theorem 1.3.9 K is an essential right ideal of R and so $M \cap K \neq 0$. Then $(M \cap K)S$ is a nonzero right ideal of S, since S is semiprime and so $l_S(S) = 0$. Then $(M \cap K)S \cap J \neq 0$. Let $0 \neq q = \sum_{i=1}^{n} k_i q_i \in (M \cap K)S \cap J$, where $k_i \in M \cap K$ and $q_i \in S$. By Lemma 3.3.1 $q_i L \subseteq R$ for some dense right ideal $L \subseteq K$ of R, $i = 1, 2 \ldots n$. obviously $0 \neq qL \in M \cap (J \cap R)$ and so $0 \neq qLK \in M \cap ((J \cap R)K)$. Therefore $(J \cap R)K$ is an essential right ideal of R.

Conversely, let $(J \cap R)K$ be an essential right ideal of R and let P be a nonzero right ideal of S. Choosing $0 \neq p \in P$ we see from Lemma 3.3.1 that $pL \subseteq K$ for some $L \subseteq K$, $L \in \mathcal{D}(R)$. Then $P \cap R \neq 0$, $(P \cap R)K \neq 0$ and hence $J \cap P \supseteq (J \cap R)K \cap (P \cap R)K \neq 0$. Thus $J \cap P \neq 0$. Therefore J is an essential right ideal of S.

Proposition 3.3.2 (Beidar [5]) Let K be a dense right ideal of a ring R and S be a subring of $Q^r_{max}(R)$ such that $K \subseteq S$. Then $Q^r_{max}(S) = Q^r_{max}(R)$.

Proof We verify the four properties of Theorem 3.2 7, since $S \subseteq Q^r_{max}(R)$, (i) holds Let $q \in Q^r_{max}(R)$. By Lemma 3 3.1 $qI \subseteq K$ for some $I \in \mathcal{D}(R)$, $I \subseteq K$. According to Lemma 3.3.2 $IS \in \mathcal{D}(S)$ and we have $qIS \subseteq KS \subseteq S$, thus proving (ii). Next suppose $qJ = 0$ for some $q \in Q^r_{max}(R)$, $J \in \mathcal{D}(S)$. By Lemma 3.3.2(ii) $(J \cap R)K \in \mathcal{D}(R)$ whence $q = 0$ and so (iii) is proved. Finally suppose we are given $f : J_S \to S_S$, $J \in \mathcal{D}(S)$. Setting

$$L = \{x \in (J \cap R)K \mid f(x) \in R\}$$

We shall show that $L \in \mathcal{D}(R)$ and $f : L \to R$ is a homomorphism of right R-modules. Note that $(J \cap R)K \in \mathcal{D}(R)$ by Lemma 3.3 2(ii) Since $K \subseteq S$, f is a homomorphism of right K-modules and so by Proposition 3.3 1(vii) f is a homomorphism of right R modules It follows from Proposition 3.3.1 (i) that $L = f^{-1}((J \cap R)K) \in \mathcal{D}(R)$. Thus there exists $q \in Q^r_{max}(R)$ such that $f(x) = qx$ for all $x \in L$, We claim that $f(z) = qz$ for all $z \in J$. Indeed by Lemma 3.3.2 $LS \in \mathcal{D}(S)$. Clearly $LS \subseteq J$ and $f(z) = qz$ for all $z \in LS$ Given any $z \in J$ and $(z \; LS)_S$ we have

$$(f(z) - qz)s = f(z)s - qzs = f(zs) - qzs = qzs - qzs = 0$$

Since $(z : LS)_S \in \mathcal{D}(S)$, we conclude that $f(z) = qz$ for all $z \in J$, and (vi) has thereby been shown.

Theorem 3.3.1 (Beidar [5]) Let R be a semiprime ring and $Q = Q^r_{max}(R)$. Then $Q^r_{max}(Q) = Q$.

Corollary 3.3.2 Let R be a semiprime ring, I an ideal of R and $J = l_R(I)$. Then $Q^r_{max}(R) = Q^r_{max}(I) \oplus Q^r_{max}(J)$

Lemma 3.3.3 Let R be a semiprime ring and K be an essential right ideal of R. If $r \in R$, then

(i) $(r : K)_R$ is an essential right ideal of R.

(*ii*) $\mathcal{Z}(R_R)$ is an ideal of R.

(*iii*) $\mathcal{Z}(R_R) = 0$ if and only if every essential right ideal is dense.

(*iv*) For any subring $R \subseteq S \subseteq Q^r_{max}(R)$, $\mathcal{Z}(R_R) = R \cap \mathcal{Z}(S_S)$.

Proof (i) Let $L \neq 0$ be a right ideal of R. If $rL = 0$, then $L \subseteq (r : K)$ and hence $0 \neq L = L \cap (r : K)$. Suppose that $rL = 0$. Since rL is a right ideal of R, $rL \cap K \neq 0$. But $rL \cap K = r[L \cap (r : K)]$. Therefore $L \cap (r : K) \neq 0$ and $(r : K)$ is essential.

(ii) Let $r_1, r_2 \in \mathcal{Z}(R_R)$ and $x \in R$. Since $r_R(r_1 - r_2) \supseteq r_R(r_1) \cap r_R(r_2)$ and $r_R(r_1) \cap r_R(r_2)$ is an essential right ideal, $r_R(r_1 - r_2)$ is essential as well. Hence $r_1 - r_2 \in \mathcal{Z}(R_R)$. Further as $r_R(xr_1) \supseteq r_R(r_1)$, $xr \in \mathcal{Z}(R_R)$. By the above result the right ideal $(x : r_R(r_1))$ is essential. From $r_R(r_1x) \supseteq (x : r_R(r_1))$ it follows that $r_1x \in \mathcal{Z}(R_R)$. Therefore $\mathcal{Z}(R_R)$ is an ideal of R.

(iii) Suppose that $\mathcal{Z}(R_R) = 0$. Let J be an essential right ideal of R. Taking into account (i), we get $l_R((a : J)) = 0$ for all $a \in R$. By Corollary 3.3.1 we then have $J \in \mathcal{D}$. The converse statement follows Proposition 3.3.1(v).

(iv) Note that $r_R(x) = r_S(x) \cap R$ for all $x \in R$ and so by Lemma 3.3.2 $r_R(x)$ is an essential right ideal of R, if and only if $r_S(x)$ is an essential right ideal of S. Hence $\mathcal{Z}(R_R) = \mathcal{Z}(S_S) \cap R$.

Lemma 3.3.4 Let R be a semiprime ring, $Q = Q^r_{max}(R)$ and K be a submodule of the right R-module Q. Suppose that $\alpha : K \to Q$ is a homomorphism of right R-modules. Then

(*i*) The rule $\widetilde{\alpha}(\sum_{i=1}^{n} k_i q_i) = \sum_{i=1}^{n} \alpha(k_i) q_i$ where $k_i \in K$ and $q_i \in Q$ defines a homomorphism of right Q-modules $\widetilde{\alpha} : KQ \to Q$.

(*ii*) If K is a right ideal of the ring Q, then α is a homomorphism of right Q-modules.

Proof (i) It is enough to check that $\widetilde{\alpha}$ is well defined. Indeed, let $\sum\limits_{i=1}^{n} k_i q_i = 0$ where $k_i \in K$, $q_i \in Q$. By Lemma 3.3.1 there exists a dense right ideal L of R such that $q_i L \subseteq R$ for all i. For any $x \in L$ we have

$$[\sum_{i=1}^{n} \alpha(k_i)q_i]x = \sum_{i=1}^{n} \alpha(k_i)(q_i x) = \alpha(\sum_{i=1}^{n} k_i q_i x) = 0.$$

Therefore $\sum\limits_{i=1}^{n} \alpha(k_i)q_i = 0$ and $\widetilde{\alpha}$ is well defined.

(ii) If K is a right ideal of the ring Q, then $\alpha = \widetilde{\alpha}$ which means that α is a homomorphism of right Q-modules.

Theorem 3.3.2 (Beidar [5]) Let R be a semiprime ring and $Q = Q^r_{max}(R)$. Then the following conditions are equivalent:

(i) Q is a von-Neumann regular ring.

(ii) $\mathcal{Z}(R_R) = 0$.

(iii) Furthermore, if the above conditions are fulfilled, then Q is an injective right R-module and Q-module.

Proof Setting $Q = Q^r_{max}(R)$, we suppose that Q is von Neumann regular. Let $0 \neq q \in Q$. Then $qxq = q$ for some $x \in Q$. Obviously $r_Q(xq) = r_Q(q)$ and $(xq)^2 = xq$. Hence $r_Q(xq) = (1 - xq)Q$. Since $(1 - xq)Q \cap xqQ = 0$, the right ideal $(1 - xq)Q$ is not essential. Therefore $\mathcal{Z}(Q_Q) = 0$. By Lemma 3.3.3, $\mathcal{Z}(R_R) = 0$.

Conversely, let $\mathcal{Z}(R_R) = 0$. Then by Lemma 3.3.3, the set $\mathcal{D}(R)$ coincides with the set of all essential right ideals of R. Let $q = [f; J] \in Q = Q^r_{max}(R)$. We set $K = Ker(f)$. Choosing L to be a right ideal of R maximal with respect to the properties $L \subseteq J$ and $L \cap K = 0$, we note that $L \cong qL$. One can easily check that $K + L$ is an essential right ideal of R and hence $k + L \in \mathcal{D}(R)$. Now we choose M to be a right ideal of R maximal with respect to the property $M \cap qL = 0$. It is well known that $M \oplus qL$ is an essential right ideal of R. Hence $M \oplus qL \in \mathcal{D}(R)$. Define the mapping $g : M \oplus qL \to L$ by the rule $g(m + ql) = l$ for all $m \in M, l \in L$. Clearly $p = [g; M \oplus qL] \in Q$ and $fgf(k + l) = f(k + l)$ for all $k \in K$ and $l \in L$. Therefore $qpq = q$ and Q is von Neumann regular.

We show now that Q is an injective right R-module. Let K be a submodule of the right R-module Q and $\alpha : K \to Q$ a homomorphism of right R-modules. According to Lemma 3.3.4 we can assume that K is a right ideal of the ring Q and α is a homomorphism of right Q-modules. Choosing L to be a right ideal of Q maximal with respect to the property $L \cap K = 0$, we extend α up to homomorphism $\widetilde{\alpha} : K + L \to Q$ by the rule $\widetilde{\alpha}(k+l) = \alpha(k)$ for all $k \in K$, $l \in L$. Clearly $K + L$ is an essential right ideal of Q. Since $\mathcal{Z}(Q_Q) \cap R = \mathcal{Z}(R_R) = 0$, we infer that $\mathcal{Z}(R_R) = 0$ (by Lemma 3.3.3 and Theorem 3.2.7). Then according to Lemma 3.3.3(iii), $K + L$ is a dense right ideal of Q. Hence $[\widetilde{\alpha}, K + L] \in Q^r_{max}(Q)$. Since $Q^r_{max}(Q) = Q$ by Theorem 3.3.1, there exists an element $q \in Q$ such that $\widetilde{\alpha} = l_q$ where l_q is left multiplication by q. Thus we have extended the mapping $\alpha : K \to Q$ up to an endomorphism of Q_Q. Applying this observation to the case $K \subseteq R$ we conclude that Q_R is an injective module. On the other hand, applying this observation to the case $K_Q \subseteq Q_Q$ we infer that Q_Q is an injective Q module.

3.4 Martindale-Amitsur rings of quotients

After discussing maximal rings of quotients, it is natural to include an intoduction to the idea of Martindale rings of quotients. This kind of ring of quotients was introduced by Martindale [51] for prime rings in 1969 and by Amitsur [3] for semiprime rings in 1972. A more precise term for such ring of quotients would have been the Martindale-Amitsur rings of quotients as, for instance, in the book Rowen [63]. In the interest of brevity, however, we shall refer to them simply as Martindale's rings of quotients.

Definition 3.4.1 (Martindale ring of quotients) Let R be a prime ring and consider the set of all left R-module functions $f : A_R \to R_R$, where A ranges over all nonzero two sided ideals of R. Two such functions are said to be equivalent if they agree on their common domain which is a nonzero ideal, since R is prime . That is an equivalence relation. Let $\widetilde{f}$ denote the equivalence class of f and let $Q_l = Q_l(R)$ be the set of all such equivalence classes. The arithmetic in Q_l is defined as a fairly obvious manner. Suppose $f : A_R \to R_R$ and $g : B_R \to R_R$ are given. Then $\widetilde{f} \times \widetilde{g}$ is the class of $f \times g : (A \cap B)_R \to R_R$ and $\widetilde{f}\,\widetilde{g}$ is the class of the composite function

$fg : (BA)_R \to R_R$. The ring axioms are satisfied, so Q_l is a ring and is called a left Martindale ring of quotients of R.

One can of course define $Q_r = Q_r(R)$, the right Martindale ring of quotients of R in a similar manner. It is obtained from the set of all right R-module homomorphism $g : B_R \to R_R$ with nonzero ideal B of R.

Now we proceed to describe this construction for semiprime rings.
Let R be a semiprime ring and $\mathcal{I} = \mathcal{I}(R) = \{I \mid I$ is an ideal of R and l(I)=0$\}$.
We note that $\mathcal{I}$ is closed under products and finite intersections. we also mention that any $I \in \mathcal{I}$ is dense and essential as a right (or left) ideal and accordingly we shall call such ideals dense. Consider the set

$$\mathcal{T} = \{(f; J)/J \in \mathcal{I}, f : J_R \to R_R\}$$

and define $(f; J) \sim (g; K)$ if there exists $L \subseteq J \cap K$ such that $L \in \mathcal{I}$ and $f = g$ on L. We let $\{f; J\}$ denote the equivalence class determined by $(f; J) \in \mathcal{T}$. We now define addition and multiplication of equivalence classes as follows.

$$\{f; J\} + \{g; K\} = \{f + g; KJ\} \tag{3.4.1}$$

$$\{f; J\}\{g; K\} = \{fg; KJ\} \tag{3.4.2}$$

We will only show that multiplication is well defined. First of all we note that $KJ \in \mathcal{I}$ whenever $K, J \in \mathcal{I}$. Indeeed, let $rKJ = 0$ for some $r \in R$. Then $rK \subseteq l(J) = 0$ and so $rK = 0$. Hence $r \in l(K) = 0$, $r = 0$ and $KJ \in \mathcal{I}$. Further $g(KJ) = g(K)J \subseteq J$and so the composition fg is well defined on KJ. If $(f_1; J_1) \sim (f_2; J_2)$ and $(g_1; K_1) \sim (g_2; K_2)$ we may find $L \in \mathcal{I}$ such that $L \subseteq J_1 \cap J_2$, $f_1 = f_2$ on L and $M \in \mathcal{I}$ such that $M \subseteq K_1 \cap K_2$, $g_1 = g_2$ on M.
Set $N = ML$ and let $x \in N$. Then $N \in \mathcal{I}$ and

$$f_1 g_1(x) = f_1(g_1(x)) = f_1(g_2(x)) = f_2(g_2(x))$$

(noting that $g_1(x) = g_2(x) \in L$). Thus (3.4.2) is well defined. We can verify that the ring axioms hold, and so the construction is complete. We shall denote the ring constructed above by $Q_r = Q_r(R)$ and shall call it the **two-sided right ring of quotients of** R.

Analogously one can construct the two sided left-ring of quotients of R.

Proposition 3.4.1 Let R be a semiprime ring Then $Q_r(R)$ satisfies:

(i) R is a subring of Q_r.

(ii) For all $q \in Q_r$ there exists $J \in \mathcal{I}$ such that $qJ \subseteq R$

(iii) For all $q \in Q_r$ and $J \in \mathcal{I}$, $qJ = 0$ if and only if $q = 0$.

(iv) For any ideal $J \in \mathcal{I}(R)$ and $f : J_R \to R_R$ there exists $q \in Q_r$ such that $f(x) = qx$ for all $x \in J$.

Furthermore, properties $(i) - (iv)$ characterize ring $Q_r(R)$ up to isomorphism.

Proof We have only to prove the last statement. Let Q be a ring satisfying $(i) - (iv)$. For $q \in Q$, using (i) and (ii), we define $q^{\alpha} = \{f; J\}$ where $qJ \subseteq R$ $J \in \mathcal{I}$ and $f(x) = qx$ for all $x \in J$. One readily checks that $\alpha : Q \to Q_r$ is a ring homomorphism. By (iii) α is an injection and by (iv) α is surjective and so α is a ring isomorphism.

Proposition 3.4.2 Given a semiprime ring R, there exists a unique ring monomorphism $\sigma : Q_r(R) \to Q^r_{max}(R)$ such that $r^{\sigma} = r$ for all $r \in R$ Further,

$Im(\sigma) = \{q \in Q^r_{max}(R) \mid qJ \subseteq R \text{ for some } J \in \mathcal{I}\}$

Proof Define the mapping $\sigma : Q_r \to Q^r_{max}$ by the rule $\{f; J\}^{\sigma} = [f; J]$ for all $\{f; J\} \in Q_r$ It follows directly from the definitions of $\sim$ and $\simeq$ that σ is well defined Obviously $r^{\sigma} = \{l_r; R\}^{\sigma} = [l_r; R] = r$ for all $r \in R$ Let $\{f; J\}, \{g; K\} \in Q_r$. Since $KJ \subseteq K \cap J$ and $KJ \in \mathcal{I}$, we have

$(\{f; J\} + \{g; K\})^{\sigma} = \{f + g; KJ\}^{\sigma} = [f + g; KJ] = [f; J] + [g; K]$

and σ is additive. Noting tht $KJ \subseteq g^{-1}(J)$, one easily checks that σ preserve products. If $\{f; J\}^{\sigma} = 0$, then $f(L) = 0$ for some dense right ideal $L \subseteq J$. Then by Remark 3.3.1 $f = 0$ and therefore σ is a monomorphism If $\sigma' : Q_r(R) \to Q^r_{max}(R)$ is another ring monomorphism such that $r^{\sigma'} = r$ for all $r \in R$, then for every $q \in Q_r(R)$ and $x \in (q : R)_R$ we have

$(q^{\sigma} - q^{\sigma'})x = q^{\sigma}x^{\sigma} - q^{\sigma'}x^{\sigma'} = (qx)^{\sigma} - (qx)^{\sigma'} = qx - qx = 0$

and so $q^\sigma = q^{\sigma'}$ for all $q \in Q_r(R)$ thus proving the uniqueness. We set

$$Q = \{q \in Q^r_{max}(R) \mid qJ \subseteq R \text{ for some } J \in \mathcal{I}\}.$$

Clearly $Im(\sigma) \subseteq Q$. Let $q \in Q$. Then $qJ \subseteq R$ for some $J \in \mathcal{I}$. We define $f : J \to R$ by the rule $f(x) = qx$ for all $x \in J$. Setting $q' = \{f; J\}^\sigma$, we note that $qa = q'a$ for all $a \in J$. Applying Theorem 3.3.1(iii) we infer that $q = q'$ and thus $Q = Im(\sigma)$.

In what follows we shall identify Q_r with Q via σ. We set

$$Q_S = \{q \in Q^r_{max}(R) \mid qJ \cup Jq \subseteq R \text{ for some } J \in \mathcal{I}\}$$

One can easily check that Q_S is a subring of Q_r. We shall call it the **symmetric ring of quotients** of R. As noted by Passman (Proposition 1.4, [56]) Q_S may be characterized by four properties anologous to those which characterize Q^r_{max}.

Proposition 3.4.3 Let R be a semiprime ring. Then $Q_S(R)$ satisfies:

(*i*) R is a subring of Q_S.

(*ii*) For all $q \in Q_S$ there exists $J \in \mathcal{I}$ such that $qJ \cup Jq \subseteq R$.

(*iii*) For all $q \in Q_S$ and $J \in \mathcal{I}$, $qJ = 0$(or $Jq = 0$) if and only if $q = 0$.

(*iv*) Given $J \in \mathcal{I}$, $f : J_R \to R_R$ and $g :_R J \to_R R$ such that $xf(y) = g(x)y$ for all $x, y \in J$, there exists $q \in Q$ such that $qx = f(x)$, $xq = g(x)$ for all $x \in J$. Furthermore, Properties (i) – (iv) characterize ring $Q_S(R)$ up to isomorphism.

Proof We can verify that Q_S enjoys the properties (i) – (iv). Now assume that Q is a ring satisfying (i) – (iv). We define a map $Q \to Q_{mr}$ by the rule $q \mapsto q' = [f; J]$, where J is given by (ii) and f is defined by $f(x) = qx$ for all $x \in J$. Again by (ii) one shows that for all $a \in J$

$$[l_a; R][f; J] = [l_a f; J] = [l_{aq}; J]$$

ie $aq' \in R$, whence $q' \in Q_S$. It is straightforward to show that $q \mapsto q'$ is an injection follows from property (iii).

Finally given $p \in Q_S$ we have $pJ + Jp \subseteq R$ for some $J \in \mathcal{I}$. We then define $f : J_R \to R_R$ by $f(x) = px$ for all $x \in J$ and $g :_R J \to_R R$ by $g(x) = xp$ for all $x \in J$. Thus $g(x)y = (xp)y = x(py) = xf(y)$ for all $x, y \in J$, and so by property (iv) there exists $q \in Q$ such that $q(x) = f(x)$, $xq = g(x)$ for all $x \in J$. Clearly

$q' = p$ and so $q \mapsto q'$ is surjective. The Proof of Proposition 3.4.3 is now complete.

Remark 3.4.1 Let R be a semiprime ring. Then

$$Z(Q_S) = C = Z(Q^r_{max}(R)) = \{q \in Q^r_{max}(R) \mid qr = rq \text{ for all } r \in R\}$$

Proof If $c \in Z(Q^r_{max}(R))$, $x \in (c : R)_R$ and $r \in R$, then $c(rx) = r(cx) \in R$, $rx \in (c : R)_R$, and so $J = (c : R)_R$ is a dense ideal of R. Since $Jc = cJ \subseteq R$, $c \in Q_S$ and $Z(Q^r_{max}) \subseteq Z(Q_S)$. According to Proposition 3.3.2, $Q^r_{max}(Q_S) = Q^r_{max}(R)$. Therefore $Z(Q_S) \subseteq Z(Q^r_{max})$ and $Z(Q_S) = Z(Q^r_{max})$. Analogously one can show that $Z(Q_r) = Z(Q^r_{max})$.

If $q \in Q^r_{max}$ and $qr = rq$ for all $r \in R$, then $(qx - xq)r = q(xr) - xqr = xrq - xrq = 0$ for all $x \in Q^r_{max}$, $r \in (x : R)_R$. Thus $q \in C$.

Remark 3.4.2

(i) Let the semiprime ring R be right duo i.e. such that right ideals in R are ideals. Then $F = F(R)$ is the family of all dense right ideals. In this case, $Q_r(R)$ boils down to the entire maximal ring of quotients $Q^r_{max}(R)$. This is the case, for instance, if R is any commutative ring, then we also have $Q_S(R) = Q^r_{max}(R)$.

(ii) If R is any simple ring, then clearly $F(R)$ consists of a single ideal that is R. In this case $Q_r(R) = R$ and also $Q_S(R) = R$.

CHAPTER IV

Generalized polynomial identities with coefficients in rings of quotients

4.1 Introduction

A generalized polynomial identity (GPI) of an algebra A over a field F is a polynomial expression f in noncommuting indeterminates and fixed coefficients from A between the indeterminates such that f vanishes upon all substitutions by elements of A. It is a natural extension of the notion of a polynomial identity (PI) in which the coefficients come from the base field.

PI-theory began with a paper of Kaplansky [40] on primitive rings appeared in 1948. The theory of GPI was initiated by Amitsur in 1965 with his fundamental paper on primitive GPI rings. In 1969 Martindale [51] extended Amitsur's work to prime GPI rings. A key notion in making the transition to a prime ring R was that of the extended centroid C and the resulting central closure $S = RC$, it becoming clear that C (rather than the field of fractions of the centroid) was the proper field of scalars in case of prime rings.

Section 4.2 contains some basic definitions and important results of Rowen [63] regarding polynomial identities.

Section 4.3 is devoted to the study of generalized polynomial identities over centroid C of the Martindale ring of quotients. The main result of the section is of Martindale [51] which states that the central closure $S = RC$ of a prime ring R satisfies a generalized polynomial identity over C if and only if for a nonzero idempotent e in R, S contains a minimal right ideal eS and eSe is a finite dimensional division algebra over C. This result generalizes Kaplansky's Theorem, Amitsur's Theorem and Posner's Theorem. Let T be a nonzero R-submodule of the Martindale ring of quotients Q. T satisfies a Q-generalized polynomial identity (Q-GPI) if for some $f(x_1, x_2, ..., x_n)$
$\in Q_C < X > - \{0\}$, $f(t_1, t_2, ..., t_n) = 0$ for all substitutions of $t_i \in T$ for x_i. This notion generalizes the usual situation when R satisfies a GPI, which means a Q-GPI with coefficients in $RC + C$, the central closure of ring R.

Lansky [49] proved that the set of multilinear and homogeneous Q-GPI is the same for any R-submodule of Q and that if R satisfies a Q-GPI, then it satisfies one having all its coefficients in R.

Section 4 4 contains some Theorems of Chuang [14] which are generalizations of the above result for rational submodules of Utumi quotient rings.

4.2 Polynomial identities

Definition 4.2.1 (Polynomial identity) A polynomial f is called an identity of a ring R if $f(R) = 0$. An identity f is a polynomial identity if one of the monomials of f of highest degree has coefficient 1. R is a PI algebra if R satisfies a polynomial identity and R is a PI-ring if R satisfies a polynomial identity with $C = \mathbb{Z}$, the ring of integers.

Remark 4.2.1 If f is a polynomial and σ is a C-algebra automorphism of a ring R, then $f(\sigma r_1, .., \sigma r_m) = \sigma f(r_1, r_2, \quad r_m)$. Consequently $f(R)$ and $f(R)^+$(additive subgroup of R generated by $f(R)$) are invariant under all automorphisms of R.

Definition 4.2.2 (t-normal) A polynomial f is said to be linear in X_i if X_i occurs exactly once (of degree 1) in every monomial of f. f is called t-linear if f is linear in $X_1, X_2, .. X_t$. f is called t-alternating if $X_i \mapsto X_j = 0$ for all $1 \leq i < j \leq t$. A polynomial which is t-linear and t-alternating is said to be t-normal.

Example 4.2.1 $[X_1, X_2] = X_1X_2 - X_2X_1$ is 2-normal.

Remark 4.2.2 If $f(X_1, X_2.., X_m)$ is t-normal and R is spanned by fewer than t elements over a commutative ring C, then f is an identity of R.

Definition 4.2.3 (Capelli polynomial) The Capelli polynomial is defined as
$$C_{2t}(X_1, ..., X_{2t}) = \sum_{\pi \in Sym(t)} (sg\pi) X_{\pi 1} X_{t+1} X_{\pi 2} X_{t+2} ... X_{\pi(t-1)} X_{2t-1} X_{\pi t} X_{2t}$$

and the standard polynomial denoted by

$$S_t(X_1,...,X_t) = C_{2t}(X_1,...,X_t,1,..,1) = \sum_{\pi\in Sym(t)} (sg\pi)X_{\pi 1} \cdots X_{\pi t}$$

Remark 4.2.3 If $f(X_1,..,X_m)$ is t-linear, then

$$f(\sum c_{i1}r_{i1},...,\sum c_{it}r_{it},r_{t+1},..,r_m)$$
$$= \sum_{i_1,\dots,i_t} c_{i_1 1}...c_{i_t t} f(r_{i_1 1},..,r_{i_t t},r_{t+1},..,r_m)$$

for all c_{ij} in C and r_{ij} in R. In particular, if B spans R as C-module, then $f(R^{(m)}) = f(B^{(t)} \times R^{(m-t)})$.

Proposition 4.2.1 C_{2t} and S_t are t-normal and thus are identities of any C-algebra spanned by $< t$ elements.

Proof The terms in $C_{2t}(X_i \mapsto X_j)$ (or $S_t(X_i \mapsto X_j)$) subdivide into pairs of the same monomials appearing with opposite sign (one corresponding to a permutation π and the other corresponding to π composed with the transposition (ij) and thus having opposite sign), so the polynomial is sent to 0.

We say R is an **integral of bounded degree** n if every element of R is integral of degree $\leq n$ over C, a commutative ring, with n minimal such.

Example 4.2.2 If R is integral of bounded degree $\leq n$ and $f(X_1,...,X_m)$ is n-normal, then $S_n([X_1^n,X_2],...,[X_1,X_2])$ is a 2-variable identify of R.

Remark 4.2.4

(i) If f is an identity of R then f is also an identity of every homomorphic image of R and of every subalgebra of R.

(ii) If f is an identity of R_i for each $i \in I$, then f is an identity of $\Pi\{R_i : i \in I\}$, and thus of any subdirect product of the R_i.

Definition 4.2.4 (Multilinear polynomial) A polynomial $f(X_1,...,X_m)$ is multilinear if X_i has degree 1 in each monomial of f, for each $1 \leq i \leq m$.

Proposition 4.2.2 Any multilinear identity f of R is an identity of each central extention R' of R.

Proof If $R' = RZ$, then Remark 4.2.3 shows $f(R') = f(R)Z = 0$.

Definition 4.2.5 (Central Polynomial) $f(X_1, X_2..., X_m)$ is called a central polynomial for R, (R-central) if $0 \neq f(R) \subseteq Z(R)$, that is if f is not an identity of R but $[X_{m+1}, f]$ is an identity of R.

Proposition 4.2.3 (Rowen [63]) For $t = n^2$, the Capelli polynomial C_{2t} is not an identity of $R = M_n(H)$ for any commutative ring H (although $C_{2(t+1)}$ is an identity). Infact $C_{2t}(R)^+ = R$.

Proof Order the matric units $[e_{ij} : 1 \leq i, j \leq n]$ lexicographically on the subscripts, that is, $e_{11} < e_{12} < ... < e_{1n} < e_{21} < ... < e_{nn}$ and write r_k for the k-th matric unit on this list. Let us evaluate $C_{2t}(r_1, ...r_t, r_1, . .r_t)$. Taking π in Sym(t) let $a_\pi = r_{\pi 1} r_1 r_{\pi 2} r_2 ... r_{\pi t} r_t$. Then $r_1 r_{\pi 2} r_2 = e_{11} r_{\pi 2} e_{12}$ so $a = 0$ unless $r_{\pi 2} = e_{11}$; likewise $r_2 r_{\pi 3} r_3 = e_{12} r_{\pi 3} e_{13}$ is 0 unless $r_{\pi 3} = e_{21}$. Continuing in this way we have precisely one choice of $r_{\pi 2}, ..., r_{\pi t}$ for a to be nonzero. Since e_{n1} has not yet been selected we take $r_{\pi 1} = e_{n1}$; then $a_\pi = e_{nn}$ for this particular π, and all other $a_\pi = 0$; proving $C_{2t}(r_1, ...r_t, r_1, ...r_t) = \pm e_{nn}$. By symmetry each $e_{ii} \in C_{2t}(R)$. For $i \neq j$ we have $(1 + e_{ij})^{-1} = 1 - e_{ij}$ so $(1 + e_{ij})^{-1} e_{ii}(1 + e_{ij}) = e_{ii} + e_{ij} \in C_{2t}(R)$ by Remark 4.2.1 . Hence each $e_{ij} \in C_{2t}(R)^+$, proving $C_{2t}(R)^+ = R$.

Remark 4.2.5 No polynomial $f \neq 0$ of degree $\leq 2n - 1$ is an identity of $M_n(C)$ for a commutative ring C.

Theorem 4.2.1 (Rowen [63]) There is a multilinear polynomial which is $M_n(H)$-central for every commutative ring H.

Proof Put $t = n^2$ and write

$$\sum_{i=1}^{t} C_{2t}(X_1, ..., X_{i-1}, X_{2t+1} X_i X_{2t+2}, X_{i+1}, ..., X_{2t})$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{t} h_{ij1} X_{2t+1} X_i X_{2t+2} h_{ij2}$$

for suitable m and multilinear monomials $h_{ij1}, h_{ij2}, (in\ X_1, ..., X_{i-1}, X_{i+1}, ..., X_{2t})$. Pick arbitrary $a, b, r_1, .. r_{2t}$ in $M_n(H)$; viewing $M_n(H)$ as an n^2-dimensional module over H having base e_{ij}, $1 \leq i, j \leq n$, we define the map $T : M_n(H) \to M_n(H)$ given by $Tx = axb$. Writing $a = (a_{ij})$ and $b = (b_{ij})$ we have $Te_{ij} = \sum_{u,v=1}^{n} a_{ui} b_{jv} e_{uv}$, whose coefficient of e_{ij} is $a_{ii} b_{jj}$. Hence as $n^2 \times n^2$ matrix, T has trace $\sum_{u,v=1}^{n} a_{ii} b_{jj} = tr(a)tr(b)$ so by Theorem 1.3.12 putting $w_{iju} = h_{iju}(r_1, ..., r_{i-1}, r_{i+1}, ..., r_{2t})$ we have $tr(a)tr(b)C_{2t}(r_1, .., r_{2t}) = \sum_{i=1}^{t} C_{2t}(r_1, ...r_{i-1}, ar_i b, r_{i+1}, ...r_{2t}) = \sum_{j=1}^{m} \sum_{i=1}^{t} w_{ij1} ar_i b w_{ij2}$.
Put $w = C_{2t}(r_1, .. r_{2t})$. Taking traces of both sides we get
$tr(tr(a)tr(w)b) = tr(a)tr(w)tr(b) = tr(tr(a)tr(b)w) = \sum_{i,j} tr(w_{ij1} ar_i b w_{ij2}) = \sum_{i,j} tr(w_{ij2} w_{ij1} ar_i b) = tr(\sum_{i,j} w_{ij2} w_{ij1} ar_i)b)$,
implying $tr((tr(a)tr(w) - \sum_{i,j} w_{ij2} w_{ij1} ar_i)b) = 0$ for all b; nondegeneracy of the trace yields $tr(a)tr(w) - \sum_{i,j} w_{ij2} w_{ij1} ar_i = 0$, that is, $\sum w_{ij2} w_{ij1} ar_i$ equals the scalar $tr(a)tr(w)$. By proposition 4.2.2 we can pick $r_1, .. , r_{2t}$ such that $tr(a)tr(w) \neq 0$. Thus we define $f = \sum_{j=1}^{m} \sum_{i=1}^{t} h_{ij2} h_{ij1} X_{2t+1} X_i$, which takes only scalar values and is not an identity, that is f is $M_n(H)$-central.

Remark 4.2.6 $g_n = f(C_{2n^2}(X_1, ..., X_{2n^2}) X_{2n^2+1}, X_{2n^2+2}, ..., X_{2n^2+2t+1})$, where $f(X_1, ..., X_{2t+1})$ is the central polynomial for $n \times n$ matrices.

Corollary 4.2.1 g_n is an n^2-normal polynomial which is $M_n(C)$-central for all commutative ring C.

Definition 4.2.6 (PI degree) R is said to have PI-degree n if R satisfies all multilinear identities of $M_n(Z)$ and g_n.

Theorem 4.2.2 (Kaplansky [40]) Suppose R is a primitive ring satisfying a polynomial identity f of degree d. Then R has some PI degree $n \leq [d/2]$, and $R \approx M_t(D)$

for a division ring D (unique up to isomorphism) with $n^2 = [R : Z(R)] = t^2[D : Z(D)]$

Proof We may assume f is multilinear and $X_d...X_1$ has nonzero coefficient α in f. Let M be a faithful simple R-module and $D = End_R M$. We claim $R \approx M_t(D)$for some $t \leq d$. Otherwise, taking any x_1 in M we take r_1 such that $r_1 x_1 \notin x_1 D$ and put $x_2 = r_1 x_1$; inductively, given $r_1, \ldots, r_{i-1}$ and $x_1, \ldots, x_{i-1}$, take r_i such that $r_i x_j = 0$ for all $j < i$ and $r_i x_i \notin \sum_{j<1} x_j D$ and put $x_i = r_i x_i$. Then $f(r_1, \ldots, r_d)x_1 = \alpha r_d...r_1 x_1 = \alpha x_d \neq 0$.

Let $F = Z(D) = Z(R)$ and take an algebraically closed field K of cardinality $> 1 + [R . F]$ (possibly infinite). Then $R_1 = R \otimes_F K$ is a simple K-algebra and satisfies the identity by Proposition 4.2.2 so as above $R_1 \approx M_n(D_1)$ for some n and some K-division algebra D_1, with $[D_1 : K] \leq [R_1 : K] = [R : F] < K - 1$. Hence $D_1 = K$ by Theorem 1.3.13 so $R_1 \approx M_n(K)$ and $n \leq [d/2]$ by Remark 4.2.5. But then $n = [R : F] = t^2[D : F]$ and clearly R has PI-degree n.

In order to utilize the above result most effectively Rowen [63] proved that R is embeddible in $n \times n$ matrices if there is an injection from R into $\prod_{k<n} M_k(H_k)$ where H_k is commutative.

Theorem 4.2.3 If R satisfies a polynomial identity of degree d, then $R/N(R)$ is embeddible in $n \times n$ matrices for $n = [d/2]$. Infact, each H_K can be taken as a direct product of fields.

Proof By Theorem 1.3.14, we have an injection of $R/N(R)$ into R_1 with $Nil(R_1) = 0$. Multilinearizing we can pass PI to R_1. But $R_1[\lambda]$ is semiprimitive by Amitsur's Theorem 1.3.15, so can be injected into a direct product of primitive rings, each of which by Kaplansky's Theorem 4.2.2 is simple of dimension $\leq n^2$ over its center. Splitting each of these primitive components to inject it into $M_k(F_{ik})$ for a suitable field F_{ik} and $k \leq n$, we conclude by taking $H_k = \Pi_i F_{ik}$.

Theorem 4.2.4 (Rowen [63]) Suppose R satisfies a PI of degree d. If R is semiprime, then $Nil(R) = 0$. In general, for every nil weakly closed subset A of R we have $A^{[d/2]} \subseteq N(R)$.

Proof Let $\overline{A}$ be the image of A in $R/N(R)$ By Theorem 4.2.3 $R/N(R)$ is embeddible in $n \times n$ matrices for $n = [d/2]$, so $\overline{A}^n = 0$, that is , $A^n \subseteq N(R)$, proving the second assertion. In particular, $Nil(R)^n \subseteq N(R) = 0$ for R semiprime, so $Nil(R) = 0$.

Lemma 4.2.1 If f is an identity or central polynomial of $M_n(C)$ where C is a commutative ring, then f is an identity of $M_{n-1}(C)$

Proof Matrices $x_i = \sum_{i,j=1}^{n-1} c_{ij}e_{ij}$ are elements of $M_n(C)$ so $f(x_1, ..., x_m)$ is some scalar α but $\alpha e_{nn} = f(x_1, ...x_m)e_{nn} = 0$ so $\alpha = 0$.

Theorem 4.2.5 (Rowen [63]) Every semiprime PI -ring R has PI degree n for suitable n and every ideal of R intersects the center nontrivially.

Proof Let d be the degree of a PI of R; let $Z = Z(R)$and $0 \neq A$ be a proper ideal of R. We shall show R has PI degree $n \leq [d/2]$, and $A \cap Z \neq 0$.

Case-I R is semiprimitive. R is a subdirect product of primitive $\{R_i; i \in I\}$ and by kaplansky's theorem each R_i is central simple over $Z(R_i)$ of degree $n_i \leq [d/2]$. Let $\pi : R \to R_i$ denote the canonical projection and let $A_i = \pi A_i$. Each A_i is an ideal of R_i so $A_i = 0$ or $A_i = R_i$. Let $I' = \{i \in I : A_i \neq 0\}$ and $n = max\{n_i : i \in I'\}$ then $g_n(A_i) \subseteq Z(R_i)$ for $i \in I'$ and $g_n(A_i) = 0$ for $i \notin I'$ by Lemma 4.2.1. But $g_n(A_i) \neq 0$ for i such that $n_i = n$, so $0 \neq g_n(A) \subseteq A \cap Z$. Furthermore, taking $A = R$ we have each $A_i = R_i$ so $I' = I$ and g_n is R-central , proving R has PI degree n.

Case-II R is semiprime. Then $Nil(R) = 0$ by Theorem 4.2.4 so $R[\lambda]$ is semiprimitive by Amitsur's theorem . Now case I is applicable . $0 \neq A[\lambda] \cap Z[\lambda] = (A \cap Z)[\lambda]$ proving $A \cap Z \neq 0$; also $R[\lambda]$ has some PI -degree n, implying R also has PI- degree n.

Corollary 4.2.2 If R is a semiprime PI and $Z(R)$ is a field F, then R is a central simple F-algebra.

Proof Every nonzero ideal of R contains a unit in F, so R is simple and Kaplansky's theorem is applicable.

Theorem 4.2.6 (Rowen [63]) If R is a prime PI-ring and $S = Z(R) - \{0\}$, then $S^{-1}R$ is central simple over $S^{-1}Z(R)$ of degree n, where n=PI-degree (R).

Proof Let $Z = Z(R)$. Then $S^{-1}R$ is prime by (Proposition 2.12.9', [63]) and has PI-degree n. Since $S^{-1}R$ is a central extension of R . But Theorem 1.3.16 shows $Z(S^{-1}R) = S^{-1}Z$, a field, so $S^{-1}R$ is central simple by Corollary 4.2.2.

Crollary 4.2.3 If R is a semiprime PI-ring, then $\mathcal{Z}(R_R) = 0$.

Proof If L is a large left ideal of R and $Lz = 0$ for $z \in Z(R)$, then $z \in l(L) = 0$. Thus $Z(R) \cap \mathcal{Z}(R_R) = 0$, implying $\mathcal{Z}(R_R) = 0$.

Lemma 4.2.2 Let R has a PI of degree n. Then for each $r_1, ...r_{m+1}$ in R we have $g_n(r_1, ...r_m)r_{m+1} = \sum_{i=1}^{t}(-1)^{i+1}g_n(r_{m+1}, r_1...r_{i-1}, r_{i+1}, ...r_m)r_i$ where $t = n^2$.

Proof Define
$\overline{g}(X_1, ..., X_{m+1})$ as $\sum_{i=1}^{t+1}(-1)^i g_n(X_1, X_2..., X_{i-1}, X_{i+1}, ...X_{m+1})X_i$. Then $\overline{g}$ is $(t+1)$ alternating by inspection since $\overline{g}(X_i \mapsto X_j)$ has exactly two nonzero parts, which appear with opposite signs . Hence $\overline{g}$ is an identity of R, so $\overline{g}_n(r_{m+1}, r_1..., r_m) = 0$ yielding the desired equation.

Theorem 4.2.7 (Rowen [63]) Suppose R has PI-degree n. If there are elements $r_1, r_2..., r_m$ in R for which $g_n(r_1, ..., r_m) = 1$, then R is a free $Z(R)$-module with base $r_1, ...r_t$ where $t = n^2$.

Proof Let $Z = Z(R)$. By Lemma 4.2.2 we have
$r = \sum_{i=1}^{t}(-1)^{i+1}g(r, r_1, ..., r_{i-1}, r_{i+1}, ..., r)r_i \in \sum Zr_i$ for each $r \in R$, so $r_1, ..., r_t$ span R. If $\sum_{i=1}^{t} z_i r_i = 0$ for z_i in Z, then for each $j \leq t$ we have

$$0 = g_n(r_1, ..., r_{j-1}, \sum_{i=1}^{t} z_i r_i, r_{j+1}, ..., r_m) r_j$$
$$= \sum_{i=1}^{t} g_n(r_1, ..., r_{j-1}, z_i r_i, r_{j+1} ..., r_m) r_j$$
$$= g_n(r_1, ..., z_j r_j, ..., r_m) = z_j$$
proving $r_1, ..., r_t$ are independent.

Theorem 4.2.8 (Amitsur [2]) A primitive ring R satisfies a polynomial identity if and only if it is isomorphic with a dense ring of linear transformations over a division ring D which is finite over its center and R contains a linear transformation of finite rank.

4.3 Generalized polynomial identities over centroid

Definition 4.3.1 (Substitution map) Let R be a prime ring with extended centroid C and symmetric ring of quotients Q. Let X be an (infinite) set and $Q_C < X >$ is the coproduct of the C-algebra Q and the free algebra $C < X >$ over C. If P is any C-algebra with 1 containing Q, then any set theoretic map $X \to P$ can be extended uniquely to a C-algebra map $Q_C < X > \to P$ such that $q \mapsto q$, $q \in Q$. Such a map is called substitution.

Amitsur [2] introduced the notion of generalized polynomial identities in 1965.

Definition 4.3.2 (Generalized polynomial identity) Given an element $\phi = \phi(x_1, x_2, ... x_n) \in Q_C < X >$ and elements $p_1, p_2 ..., p_n \in P$, where P is any C algebra with identity containing Q, $\phi(p_1, p_2, ..., p_n)$ will denote the image of ϕ under the substitution determined by $x_i \mapsto p_i$. Let $0 \neq U$ be an additive subgroup of R . An element $\phi = \phi(x_1, x_2, ..., x_n) \in Q_C < X >$ is said to be a generalized polynomial identity on U, if $\phi(u_1, u_2 ... u_n) = 0$ for all $u_1, u_2 ... u_n \in U$. Henceforth we will use the abbreviation GPI and make statements such as "ϕ is a GPI on U" or "U is GPI".

Let R be a prime ring and let $S = RC$ be the central closure of R. Form C-universal product $S < X >$ of the C-algebra S and $C < X >$ where $X = \{x_1, x_2, ...\}$ is a noncommuting indeterminates $x_1, x_2 ..., x_n, ...$. Roughly speaking, the elements of $S < X >$ are of the form

$$f = \sum \beta_k a_{i_0} x_{j_1} a_{i_1} \ldots a_{i_{n-1}} x_{j_n} a_{i_n},$$

where $\beta_k \in C$, $a_{i_k} \in S$

Martindale [51] defined generalized polynomial identities over centroid.

Definition 4.3.3 (Generalized polynomial identity over centroid) S is said to satisfy a nontrivial generalized polynomial identity over C (S is GPI) if there is a nonzero element $f(x_1, x_2, ..., x_n)$ in $S < X >$ such that $f(x_1, s_2, ..., s_n) = 0$ for all $s_1, s_2, \ldots, s_n \in S$.

The degree of the monomial $a_0 x_{i_1} a_1 x_{i_2} \ldots a_{n-1} x_{i_n} a_n$ (all $a_i's \neq 0$) is n and the degree of an element f of $S < X >$ is the maximum degree of its monomials (assuming that a representation of f as a sum of monomials is chosen so that the degree of the monomial of highest degree is minimal). If S satisfies a generalized polynomial identity of degree n, n minimal, then the usual linearization process may be used to obtain a nontrivial generalized (homogeneous) multilinear identity of degree n in $x_1, x_2, \ldots, x_n$:

$$\sum \beta_i a_{i_0} x_{j_1} a_{i_1} \ldots a_{i_{n-1}} x_{j_n} a_n = 0$$

Proposition 4.3.1 Let $0 \neq \phi = \sum_{i=1}^{m} a_i x b_i \in QxQ$. Then

(i) For all nonzero ideals I of R, $\phi(I) \neq 0$

(ii) If $0 \neq I$ be a proper ideal of R such that $dim_C(\phi(I)C) < \infty$, then there exist nonzero elements $a, b \in R$ such that $dim_C(aRCb) < \infty$.

Proof Without loss of generality we may assume that $m \geq 1$ and that $\{a_i\}$ and $\{b_i\}$ are each C-independent sets. By (Theorem 2.3.3, [5]) there exists $\beta = \sum_k l_{u_k} r_{v_k} \in R_{(l)} R_{(r)}$ such that $a_1.\beta \neq 0$ but $a_i.\beta = 0$, $i > 1$. We set $\psi(x) = \sum_k u_k \phi(v_k x)$ and note that $\psi(x) = a'xb_1$ where $a' = a_1.\beta \neq 0$. Let $0 \neq I$ be a proper ideal of R and suppose $\phi(I) = 0$. Then $\psi(I) = 0$ whence we have the contradiction $a'Ib_1 = 0$. Part (i) has thereby been proved. Now suppose $0 \neq I$ be a proper ideal of R is such that $dim_C(\phi(I)C) < \infty$. Since $u_k\phi(v_kI) \subseteq u_k\phi(I)$, it follows that $dim_C(\psi(I)C) < \infty$ that is, $dim_C(a'Ib_1C) < \infty$. Pick $s \in I$ such that $0 \neq a = a's \in R$ and $t \in R$ such that $0 \neq b = tb_1 \in R$. As a result we see that $aRb \subseteq a'Ib_1$ and accordingly $dim_C(aRbC) < \infty$. The proof of (ii) is thereby complete.

Corollary 4.3.1 If $\phi \in QxQ$ is a GPI on some nonzero ideal I of R, then $\phi = 0$, that is there are no nonzero linear GPI's in one variable.

Proposition 4.3.2 Let $S = RC$ be the central closure of R and let $a, b \in S$ be nonzero elements such that $dim_C(aSb) < \infty$. Then the ring S has a nonzero idempotent e such that eS is a minimal right ideal of S and $dim_C(eSe) < \infty$. (In particular S is a primitive ring with nonzero socle).

Proof Without loss of generality we may assume that the elements $a, b \in S$ are such that $dim_C(aSb) \leq dim_C(uSv)$ for all nonzero $u, v \in S$. We claim that $M = aSbS$ is a minimal right ideal of S. Indeed, since S is prime and $a \neq 0 \neq b$, $M \neq 0$. Let $0 \neq z = \sum\limits_i ax_iby_i \in M$ where $x_i, y_i \in S$. Setting $u = \sum\limits_i x_iby_i$, we note that $z = au$. Further we have $auSb \subseteq aSb$ and so $auSb = aSb$ by the choice of a, b. Hence $auSbS = M$ and hence $M = auSbS \subseteq zS \subseteq M$ and $zS = M$ for all nonzero $z \in M$. Therefore M is a minimal right ideal of S. By Theorem 1.3.18 $M = eS$ for some idempotent e and eSe is a division ring. Clearly $e = \sum\limits_{i=1}^{m} au_ibv_i$. Hence $eSe \subseteq \sum\limits_{i=1}^{m} aSbv_i$ and so $dim_C(eSe) < \infty$ and the proposition is proved.

Remark 4.3.1 If $0 \neq \phi$ is a GPI on I of degree n, then there exists a nonzero multilinear GPI on I of degree $\leq n$.

We are now in a position to prove the main result of this section which is known as the prime GPI Theorem.

Theorem 4.3.1 Let R be a Prime ring with extended centroid C and central closure $S = RC$. Then there is a nonzero GPI ϕ on I for some nonzero ideal I of R if and only if S has a nonzero idempotent e such that eS is a minimal right ideal of S (hence S is primitive with nonzero socle) and eSe is a finite dimensional division algebra over C.

Proof If $dim_C(eSe) = n < \infty$ and St_{n+1} is the standard polynomial in $n+1$ variables, then

$$\phi = St_{n+1}(ex_1e, ex_2e, ...ex_{n+1}e)$$

is the required GPI.

Conversely let $0 \neq \phi$ be a GPI on some nonzero ideal I of R. By Remark 4.3.1 we may assume that $\phi = \phi(x_1, x_2, ..., x_n)$ is multilinear of degree n. Pick any C basis $\mathcal{A}$ of Q. The element ϕ, when written in terms of the monomials basis $\mathcal{M}(\mathcal{A})$, only involves a finite subset F of $\mathcal{A}$. By suitable reordering of the variables we may write $\phi = b_0x_1...b_{n-2}x_{n-1}\psi(x_n) + \sum \alpha_M M + \sum \beta_N N + \sum \gamma_P P$,
$\alpha_M, \beta_N, \gamma_P \in C$ where
(i) $0 \neq \psi(x_n) \in Qx_nQ$;
(ii) M is of the form $b'_0x_1...b'_{n-2}x_{n-1}\mathcal{X}(x_n)$, with $(b'_0...b'_{n-2}) \neq (b_0...b_{n-2})$ and $\mathcal{X}(x_n) \in Qx_nQ$;
(iii) N is of the form
$b''_0x_1...b''_{i-1}x_ib''_ix_nb''_{i+1}x_{i+1}...b''_{n-1}x_{n-1}b''_n$;
(iv) $x_1, x_2, ..., x_{n-1}$ appear in a different order in P.

By Proposition 4.3.2 we may assume that $\psi(I)C \nsubseteq V = \sum\limits_{d \in F} dC$. Choose $r \in I$ such that $\psi(r) \notin V$ and set

$$\rho(x_1, x_2, ...x_{n-1}) = \phi(x_1, x_2, ...x_{n-1}, r).$$

Let $\mathcal{A}'$ be a C-basis of Q containing $F \cup \{\psi(r)\}$ and consider ρ as being written in terms of the monomials basis $\mathcal{M}(\mathcal{A}')$ of $Q_C < X >$ induced by $\mathcal{A}'$. It is then clear that the monomial $H = b_0x_1...b_{n-2}x_{n-1}\psi(r)$ cannot be canceled by any monomials which arise from M, N, or P. For instance, N ends in $b''_n \in F$ whereas H ends in $\psi(r) \notin F$. Thus $0 \neq \rho$ is a GPI of degree $n-1$ on I and so by induction the proof is complete

The following corollary shows that the effect of a GPI carries up to the maximal right ring of quotients Q^r_{max}

Corollary 4.3.2 Let R be a prime ring with extended centroid C, central closure $S = RC$ and $Q = Q^r_{max}(R)$ Then the following conditions are equivalent

(i) There is a nonzero GPI ϕ on I for some nonzero ideal I of R

(ii) Any subring $S \subseteq H \subseteq Q$ is a primitive with nonzero socle and a nonzero idempotent $e \in H$ such that eHe is a finite dimensional division algebra over C

(iii) The ring Q is isomorphic to the complete ring of linear transformations of a right vector space over a division ring which is finite dimesional over its center

(iv) R is GPI

Proof (i) $\Rightarrow$ (ii) By Theorem 4 3 1 S has nonzero socle and (ii) then follows immediately from Theorem 1 3 20(ii)

(ii) $\Rightarrow$ (iii) By Lemma 3 4 1, $Q = Q^r_{max}(H)$ Now the statement (iii) follows from the symmetric version of (Theorem 4 3 7(viii), [5])

(iii) $\Rightarrow$ (iv) We identify the ring Q with this complete ring of linear transformations over a division ring Δ Let n be the dimension of Δ over its center and let e be an idempotent of rank 1 of the linear transformation ring Q Clearly $eQe \cong \Delta$ Hence the generalized polynomial $\phi = St_{n+1}(ex_1e, \quad , ex_{n+1}e) = St_{n+1}(ex_1, \quad , ex_{n+1})e$ where St_{n+1} is the standard polynomial in $(n+1)$ variables vanishes under all substitutions $x_i \mapsto q_i \in Q$, $i = 1, 2, \quad , n+1$ Pick any $a \in (e \quad R)_R$ such that $ea \neq 0$ Then $0 = \psi = St_{n+1}(eax_1, eax_2, \quad , eax_{n+1})ea \in Q_C < X >$ is a GPI on R The implication (iv) $\Rightarrow$ (i) is obvious The proof is thereby complete

Theorem 4.3.2 Let $a, b \in S$ such that $axb = bxa$ for all $x \in R$ (hence for all $x \in S$). Then a and b are C-dependent.

Proof Let us assume that $a \neq 0$ and $b \neq 0$. Let U be a nonzero ideal of R such that $aU \subseteq R$ and $bU \subseteq R$ and let $V = UaU$. We define a mapping $f : V \to R$ according to the rule

$$\sum_i x_i a y_i \to \sum x_i b y_i, \qquad x_i, y_i \in U.$$

Suppose $\sum\limits_i x_i a y_i = 0$. Then

$0 = br \sum x_i a y_i = \sum b(rx_i) a y_i = \sum a(rx_i) b y_i = ar \sum x_i b y_i$

Thus $(aU)R(\sum x_i b y_i) = 0$ and so $\sum x_i b y_i = 0$, since R is prime. This shows that f is well defined. f is an R-homomorphism because $f\{(xay)r\} = xbyr = f(xay)r$ for all $x, y \in U$ and $r \in R$. Let q denote the element of Q determined by f and let p be any element of Q, with $pW \subseteq R$ for some nonzero ideal W of R. For $x, y \in U$ and $w \in W$ we have $(qp)(wxay) = q\{(pw)xay\} = (pw)xby = p\{wxby\} = pq(wxay)$, showing that $(qp - pq)WV = 0$. Thus $qp = pq$ for all $p \in Q$ and so $q \in C$. In particular, $x(qa - b)y = qxay - xby = 0$ for all $x, y \in U$, yielding $V(qa - b)V = 0$. Since R is prime, we obtain $qa = b$.

Theorem 4.3.3 Let $a_1, a_2, ..., a_m$ be C- independent elements of S and let $b_1, b_2, ...b_m \in S$, with $b_1 \neq 0$. Suppose $B = \{\sum\limits_{i=1}^{m} a_i x b_i \mid x \in S\}$ is finite dimensional over C and e is a nonzero idempotent. Then

(a) $B \neq 0$,

(b) S has a minimal right ideal eS ,

(c) eSe is a finite dimensional division algebra over C.

Proof The proof is by induction on m. For $m = 1$ we have $B = aSb$ and so $B \neq 0$ results from the primeness of S. Thus there exists a C-basis $v_1, v_2, ..., v_k$ of B, $k \geq 1$ so that $axb = \sum\limits_{i=1}^{k} \lambda_i(x) v_i$ for all $x \in S$, where $\lambda_i(x) \in C$. Choose $r \in S$ such that $bra \neq 0$ and set $d = br$. Then $axd = \sum\limits_{i=1}^{k} \lambda_i(x)(v_i r)$ for all $x \in S$ and so aSd is an atmost k-dimensional algebra over C. Since $da \neq 0$ and S is prime, aSd properly contains its (nilpotent) radical N. aSd/N is a finite dimensional semisimple algebra and in particular has an identity element $\overline{u}$ From this it is well khown that aSd itself contains a nonzero idempotent f, since N is nilpotent. Then $fSf(\subseteq aSd)$ is a finite dimensional prime algebra over C and so $fSf \cong D_n$, where D is a finite dimensional division algebra over C. Choose e to be a (primitive) idempotent of fSf, so that $eSe = e(fSf)e \cong D$. Thus eSe is a finite dimensional division algebra over C, which in turn implies that eS is a minimal right ideal of S.

Next Suppose that $\sum\limits_{i=1}^{m} a_i x b_i = \sum\limits_{j=1}^{k} \lambda_j(x) v_j$ for all $x \in S$, where $m > 1$, $\{a_i\}$ independent, $b_1 \neq 0$, $\{v_j\}$ basis for M, $\lambda_j(\ ) \in C$ If $b_i = \gamma_i b_1$, $\gamma_i \in C$, $i = 2, 3, ..., m$, then we have $axb_1 = \sum\limits_{j=1}^{m} \lambda_j(x) v_j$ for all $x \in S$, where $a = a_1 + \sum\limits_{i=2}^{k} \gamma_i a_i \neq 0$. This case has already been worked out. Hence, by reordering subscripts, we may assume that b_1 and b_2 are C-independent. Multiplication on the right by tb_1, where $t \in S$, yields

$$\sum_{i=1}^{m} a_i x b_i t b_1 = \sum_{j=1}^{k} \lambda_j(x) v_j t b_1 \tag{4.3.1}$$

for all $x, t \in S$. On the other hand

$$\sum_{i=1}^{m} a_i (x b_1 t) b_i = \sum_{j=1}^{k} \lambda_j (x b_1 t) v_j \tag{4.3.2}$$

for all $x, t \in S$. Subtracting (4.3.2) from (4.3.1), we obtain

$$\sum_{i=2}^{m} a_i x (b_i t b_1 - b_1 t b_i) = \sum_{j=1}^{k} \{\lambda_j(x) v_j t b_1 - \lambda_j(x b_1 t) v_j\} \tag{4.3.3}$$

for all $x, t \in S$. By Theorem 4.3.2 there exists $t_0 \in S$ such that $b_2 t_0 b_1 - b_1 t_0 b_2 \neq 0$. Setting $b_i' = b_i t_0 b_1 - b_1 t_0 b_i$, $w_j = v_j t_0 b_1$ and $\mu_j(x) = -\lambda_j(x b_1 t_0)$ we then have $\sum\limits_{i=2}^{m} a_i x b_i' = \sum\limits_{j=1}^{k} \{\lambda_j(x) w_j + \mu_j(x) v_j\}$, with $b_2' \neq 0$. By induction the proof is now complete.

Theorem 4.3.4 (Martindale [51]) Let R be a prime ring and let $S = RC$ be the central closure of R. *Then S satisfies a generalized polynomial identity over C*

if and only if for a nonzero idempotent e in R, S contains a minimal right ideal eS (hence S is primitive) and eSe is a finite dimensional division algebra over C.

Proof If S enjoys the latter two properties, then we first note that eSe satisfies the standard identity $s(x_{i_1}, x_{i_2}, ..., x_{i_n}) = \sum_i \pm x_{i_1}x_{i_2}...x_{i_n} = 0$ if n exceeds the dimension of eSe over C. In other words S itself satisfies the generalized identity

$$\sum ex_{i_1}ex_{i_2}e...ex_{i_n}e = 0.$$

Conversely, suppose S satisfies a nontrivial generalized identity of minimal degree n. Without loss of generality we may assume this identity is homogeneous multilinear of degree n so that it has the form:

$$f(x_1, x_2, ...x_n) = \sum_{i=1}^{m} a_i x_1 f_i(x_2, ..., x_n) + g(x_1, x_2, ..., x_n) = 0$$

where $a_1, a_2, ..., a_m$ are C-independent elements of S, f_i are nonzero generalized homogeneous multilinear polynomials of degree $n-1$ and g is a sum of monomials none of which have x_1 as their first variable. If x_1 appears nontrivially as the last variable in some monomial of g then we may further break up the identity so as to look like

$$f = \sum_{i=1}^{m} a_i x_1 f_i + \sum_{i=1}^{k} g_i x_1 b_i + \sum p_i x_1 q_i = 0 \qquad (4.3.4)$$

where $b_1, b_2, ...b_k$ are C-independent elements of S, g_i is of degree $n-1$ and p_i and q_i are generalizad polynomials of positive degree. For $t \in S$, multiplication of (4.3.4) on the right by tb_1 yields

$$\sum_{i=1}^{m} a_i s_1 f_i t b_1 + \sum_{i=1}^{k} g_i s_1 b_i t b_1 + \sum p_i s_1 q_i t b_1 = 0 \qquad (4.3.5)$$

for all $s_1, s_2, ...s_n$, $t \in S$. Substitution of x_1 by $s_1 b_1 t$ in (4.3.4) leads to

$$\sum_{i=1}^{m} a_i s_1 b_1 t f_i + \sum_{i=1}^{k} g_i s_1 b_1 t b_i + \sum p_i s_1 b_1 t q_i = 0 \qquad (4.3.6)$$

for all $s_1, s_2, ...s_n, t \in S$. Subtraction of (4.3.6) from (4.3.5) gives

$$\sum_{i=1}^{m} a_i s_1 (f_i t b_1 - b_1 t f_i) + \sum_{i=2}^{k} g_i s_1 (b_i t b_1 - b_1 t b_i) + \sum p_i s_1 (q_i t b_1 - b_1 t q_i) = 0 \quad (4.3.7)$$

for all $s_1, s_2, ...s_n, t \in S$. Suppose $f_1tb_1 - b_1tf_1 = 0$ for all $s_2, ..., s_n$, $t \in S$. By Theorem 4.3.2, $f_1(s_2, ..., s_n) = \lambda(s_2, ...s_n)b_1$ for all $s_2, ...s_n \in S$, where $\lambda(s_2, ...s_n) \in C$. By the minimality of n, $f_1(r_2, r_3, ...r_n) \neq 0$ for some $r_2, r_3, r_n \in S$. Define $h(x_2) = f_1(x_2, r_3, ...r_n)$, and note that $h(x_2) \neq 0$ in $S < x >$ since $h(r_2) \neq 0$. $h(x_2)$ may be written $h(x_2) = \sum_{i=1}^{j} c_i x_2 d_i$, where $\{c_i\}$ are C-independent and the d_i are nonzero elements of S. Thus $h(x) = \mu(x)b_1$ for all $x \in S$, where $\mu(x) = \lambda(x, r_3, ...r_n) \in C$ and so by Theorem 4.3.3 we are finished.

We may therefore assume that $f_1t_0b_1 - b_1t_0f_1 \neq 0$ for some $r_2, r_3, ...r_n$, $t_0 \in S$. Setting

$f_i' = f_it_0b_1 - b_it_0f_i$, $\quad b_i' = b_it_0b_1 - b_1t_0b_i \quad$ and $\quad q_i' = q_it_0b_1 - b_1t_0q_i$

we have, in view of (4.3.7), that S satisfies

$$\sum_{i=1}^{m} a_ix_1f_i' + \sum_{i=2}^{k} g_ix_1b_i' + \sum p_ix_1q_i' = 0 \tag{4.3.8}$$

where $f_1'(r_2, ...r_n) \neq 0$. (4.3.8) is not trivial identity, since this would imply that $\sum_{i=1}^{m} a_ix_1f_i'$ were trivial, which in turn would contradict Theorem 4.3.3 by specializing $x_i = r_i$, $i = 2, 3, ..., n$. Furthermore, we make the important observation that in transforming the identity (4.3.4) to the identity (4.3.7) in no monomial has the order in which the variables $x_1, x_2, ...x_n$ appear been changed (some monomials may have disappeared).

Repetition of the above process at most k times will enable us to transform our original identity (4.3.4) into a nontrivial one of the form

$$\sum_{i=1}^{m} a_ix_1f_i(x_2, ...x_n) + g(x_1, x_2, ...x_n) = 0 \tag{4.3.9}$$

in which x_1 never appears as the last variable in any monomial of g and in which no new order of the variables is introduced in any monomial.

We may assume that $x_1, x_2, ...x_r$, $r \leq n$, are those variables which appeared first in some monomial of the original identity. Applying the preceding process to each of these variables, we may in a finite number of steps transform the original identity one of the form

$$\sum a_ix_1f_i + \sum b_ix_2g_i + ... + \sum d_ix_rh_i = 0 \tag{4.3.10}$$

in which $\{a_i\}, \{b_i\}, \ldots, \{d_i\}$ are C-independent sets in S and $f_i, g_i, \ldots, h_i$ are nonzero $n-1$ degree generalized polynomials in which none of $x_1, x_2, \ldots, x_j$ ever appear as the last variable in any monomial. Since some variable must appear last in each monomial, we conclude that $j < n$. By the minimality of n, $f_1(r_2, r_3, \ldots, r_n) \neq 0$ for some $r_2, r_3, \ldots, r_n \in S$. Let

$$f_i'(x_2, \ldots, x_{n-1}) = f_i(x_2, x_3, \ldots, x_{n-1}, r_n),$$
$$g_i'(x_1, x_3, \ldots, x_{n-1}) = g_i(x_1, x_3, \ldots, x_{n-1}, r_n), \ldots, h_i'(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1})$$
$$= h_i(x_1, \ldots, x_{j-1}, x_{j+1}, x_{n-1}, r_n)$$

We claim that

$$\sum a_i x_1 f_i' + \sum b_i x_2 g_i' + \cdots + \sum d_i x_j h_i' = 0 \qquad (4.3.11)$$

is a nontrivial identity of degree $n-1$. If (4.3.11) is trivial, then it would follow that $\sum a_i x_1 f_i'$ is trivial. Setting $c_i = f_i'(r_2, r_3, \ldots, r_n)$ we would then have $\sum a_i x c_i = 0$ for all $x \in S$, with $\{a_i\}$ C-independent and $c_1 = f(r_2, r_3, \ldots, r_n) \neq 0$. This is a contradiction to Theorem 4.3.3 and therefore (4.3.11) must be a nontrivial identity. This, however, now contradicts the minimality of n.

We first specialize to the situation where R is a primitive ring. R may be considered as an irreducible ring of endomorphisms of an additive abelian group V, so that $D = Hom_R(V, V)$ is a division ring. Let F be the center of D and set $T = RF$, a subring of Hom(V, V) with same division ring D. Clearly F is contained in the extended center C of T and conversely, the proof of Theorem 12 in [52], shows that C is contained in F. Thus $C = F$ and so the central closure of the ring T is T itself. Hence the consequence of Theorem 4.3.4 are the following.

Theorem 4.3.5 (Kaplansky [40]) Let R be a primitive ring satisfying a polynomial identity over its centroid. Then R is a finite dimensional central simple algebra.

Proof We may assume that R satisfies a homogeneous multilinear identity over its centroid Z

$$x_1 x_2 \cdots x_n + \sum_{\alpha \neq I} \alpha_i x_{i_1} x_{i_2} \cdots x_{i_n} = 0, \qquad \alpha_i \in Z, \qquad 4.3.12$$

which is also satisfied by T. If $[V : D] \geq n$, then T contains a subring which has as a homomorphic image D_n, the $n \times n$ matrices over D. (4.3.12) is therefore satisfied by D_n, but this is clearly impossible if we set $x_1 = e_{11}$, $x_2 = e_{12}$, $x_3 = e_{22}$, etc., where the e_{ij} are the usual matrix units of D_n. Hence V is finite dimensional over D. Finally, by Theorem 4.3.4, D is finite dimensional over its center F.

The primitive rings R studied by Amitsur in [2] are F-algebras (i.e $RF \subseteq R$) and so Theorem 4.3.4 directly implies

Theorem 4.3.6 (Amitsur [2]) Let R be a primitive ring such that $RF \subseteq R$, where F is the center of the associated division ring D. Then R satisfies a generalized polynomial identity over F if and only if R contains a minimal right ideal and D is finite dimensional over F.

Theorem 4.3.7 (Posner [59]) Let R be prime ring satisfying a polynomial identity over its centroid Z. Then R can be embedded as either a left or right order in its central closure $S = RC$ and S is finite dimensional central simple algebra over C

Proof We first assume that R satisfies a homogeneous multilinear identity

$$\sum \alpha_i x_{i_1} x_{i_2} ... x_{i_n} = 0 \qquad \alpha_i \in Z, \qquad 4.3.13$$

and that $S = RC$ satisfies the same identity. Furthermore, because different monomials have the variables in a different order (4.3.13) remains a nontrivial identity over C. By Theorem 4.3.4, S is in particular primitive and so, by Kaplansky's Theorem 4.3.5, S is a finite dimensional central simple algebra over C.
In order to show that R is an order in S we shall first show that every nonzero ideal U of R contains a regular element. Indeed, write $S = D_k$ and let $e_1, e_2, ..., e_k$ be the usual orthogonal idempotents in S. Write $e_i = \sum_j r_{ij} c_{ij}$, $r_{ij} \in R$, $c_{ij} \in C$. Since there are only a finite number of c_{ij}, there exists a nonzero ideal W of R such that $W \subseteq U$ and $c_{ij} W \subseteq R$ for all i, j. We can see that $e_i W^3 e_i \subseteq U$. Furthermore, $e_i W^3$ is a nonzero right ideal of R and since R is prime, $e_i W^3 e_i \neq 0$.. We now select $0 \neq u_i \in e_i W^3 e_i \subseteq U$, $i = 1, 2, ..., k$, and set $u = u_1 + u_2 + ... + u_k$. In S, u clearly has rank k and so must be a regular element of R. Now in order to show that every

element of S can be written in the form $ab^{-1}, a, b \in R$, b regular, it suffices, since $S = RC$, to show that for any finite set of elements $c_1, c_2, \ldots, c_m \in C$ we can find $a_1, a_2, \ldots, a_m \in R$, b regular, such that $c_i = a_i b^{-1}$, $i = 1, 2, \ldots, m$. Certainly there is a nonzero ideal U of R such that $c_i U \subseteq R$, $i = 1, 2, \ldots, m$. From the above argument U contains a regular element b and hence $c_i b = a_i \in R$, $i = 1, 2, \ldots, m$ or in S, $c_i = a_i b^{-1}$, $i = 1, 2, \ldots, m$. This completes the proof of Posner's Theorem.

4.4 Generalized polynomial identities having coefficients in Martindale and Utumi rings of quotients

Throughout the section U denotes the Utumi ring of quotients of the ring R and C its center (extended centroid of R). We begin with the following definitions.

Definition 4.4.1 (Rational submodule) A right R submodule M of U such that U_R is a rational extension of M_R is called a rational submodule of U.

Definition 4.4.2 (Rational ideal) A right ideal ρ of R is said to be rational ideal if R_R is a rational extension of ρ_R. Rational right ideals are also called dense right ideals.

Let T be a nonzero R-subbimodule of the Martindale ring of quotients Q. T satisfies a Q generalized polynomial identity (Q GPI) if for some $f(x_1, x_2, \ldots, x_n) \in Q_C < X > -\{0\}$, $f(t_1, t_2, \ldots, t_n) = 0$ fo all substitutions of $t_i \in T$ for x_i. This notion generalizes the usual situation when R satisfies a GPI, which means a Q-GPI with coefficients in $RC + C$, the central closure of the ring R.

Proposition 4.4.1 The Utumi quotient ring U of a ring R satisfies the following axioms.

(i) R is a subring of U.

(ii) For each $a \in U$, there exists a rational right ideal ρ of R such that $a\rho \subseteq R$.

(iii) If $a \in U$ and $a\rho = 0$ for some rational right ideal ρ of R, then $a = 0$.

(*iv*) For any rational right ideal ρ and for any right R-module homomorphism $\phi : \rho_R \to R_R$, there exists $a \in U$ such that $\phi(r) = ar$ for all $r \in \rho$.

Remark 4.4.1 For a prime ring R, a nonzero two-sided ideal is a rational right ideal of R.

In the above Proposition 4.4.1, if we consider only nonzero two-sided ideals instead of rational right ideals, then we obtain the Martindale quotient ring Q. Q can be naturally regarded as a subring of U and can be characterized as follows: For $a \in U$, $a \in Q$ if and only if $aI \subseteq R$ for some nonzero two-sided ideal I of R. Also observe that the center of U, denoted by C, coincides with the center of Q, C is the extended centroid of R.

The following theorem due to Chuang [14] is the main tool in dealing with generalized polynomials with coefficients in U.

Theorem 4.4.1 Assume that R is a prime ring and U is its Utumi quotient ring. Let N be a rational submodule of U and let $u_1, ..., u_n \in U$ be C-linearly independent. Then there exists $a \in N$ such that $u_1a, ...u_na \in N$ and such that $u_1a, ..., u_na$ are still C-linearly independent.

Lemma 4.4.1 If R has a nonzero right ideal ρ which is finite dimensional over C, then R itself is finite dimensional over C.

Proof Suppose that $\dim_C \rho C = m < \infty$. Since R acts faithfully on ρC by right multiplication, R embeds in $M_m(C)$, the ring of $m \times m$ matrices over C. So R is finite dimensional.

Lemma 4.4.2 Let V and W be two vector spaces over a field F and let $T_1, ...T_r$, be F-linearly independent linear transformations of V into W. Let B be an additive subgroup of V such that $FB = V$. Then for any finite dimensional subspace W_0 of W, either there exists $v \in B$ such that $T_1v, ..., T_rv$ are linearly independent module W_0, or there exists $S = \sum_{i=1}^{r} \alpha_i T_i \neq 0$, where $\alpha_i \in F$, which is of finite rank.

Proof of Theorem 4.4.1 Let N be the given rational submodule of U. Set $M = N \cap u_1^{-1}N \cap u_2^{-1}N \cap ... \cap u_n^{-1}N$, where $u_i^{-1}N = \{u \in U : u_i u \in N\}$. M, an intersection of finitely many rational submodules, is itself a rational submodule of U. Via left multiplication, we may regard $u_1, u_2, ...u_n$ as C-linear transformations from the C-vector space MC into the C -vector space NC. If there exists $a \in M$ such that $u_1 a, u_2 a, ...u_n a$ are C-linearly independent, then we are done. Otherwise, by Lemma 4.4.2, there exist $\alpha_1, \alpha_2, ...\alpha_n \in C$ such that $u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$ is a nonzero finite rank linear transformation on MC (via left multiplication). Set $L = M \cap u^{-1}R$. Then L is a rational submodule of M such that $0 \neq uL \subseteq R$. Since u is of finite rank on M, uL must also be finite C-dimensional. Thus R possesses a nonzero, finite C-dimensional right ideal uL. By Lemma 4.4.1, R is also finite C-dimensional and hence must be a PI-ring. Set $\rho = N \cap R$, which is a rational right ideal of R. For a prime PI-ring R, U, Q and RC are all equal to the localization of ρ at its center $Z(\rho)$ by (Theorem 2, [33]). So we can find a common denominator $\alpha \in Z(\rho)$ such that $\alpha u_1, \alpha u_2, ..., \alpha u_n \in \rho$. Since $Z(\rho) \subseteq C$, $\alpha u_1, \alpha u_2, ..., \alpha u_n$ are obviously C-linearly independent.

Let $X = \{x_1, x_2, ...\}$, the countable set consisting of the noncommuting indeterminates $x_1, x_2, ...$ Let $C\{X\}$ be the free algebra over C in the set X. Consider $U\{X\} = U *_C C\{X\}$, the free product over C of U and $C\{X\}$. Elements of $U\{X\}$ are generalized polynomials. By a nontrivial generalized polynomial, we mean a nonzero element of $U\{X\}$. An element $m \in U\{X\}$ of the form $m = q_0 y_1 q_1 y_2 q_2 ... y_n q_n$, where $\{q_0, ...q_n\} \subseteq U$ and $\{y_1, ...y_n\} \subseteq X$, is called a monomial. $q_0, ..., q_n$ are called the coefficients of m. Each $f \in U\{X\}$ can be represented as a finite sum of monomials. Such representation is certainly not unique. For a given representation of f as a sum of mononials, the coefficients of each monomial occuring in the representation are called the coefficients of f in the given representation.

Definition 4.4.3 (**V generalized**) For a subset V of U, $f \in U\{X\}$ is called a V-generalized polynomial if f has a representation with all of its coefficients in V.

Definition 4.4.4 (**B-monomial**) Let B be a set of C-independent vectors of

U. B-mononmial is defined as a monomial of the form $u_0y_1u_1y_2u_2...y_nu_n$, where $\{u_0, ..., u_n\} \subseteq B$ and $\{y_1, y_2, ..., y_n\} \subseteq X$.

In [49] Lanski proved that the set of multilinear and homogeneous Q-GPI is the same for any R-submodule of Q and that if R satisfies a Q-GPI, then it satisfies one having all its coefficients in R.

Chuang further generalized the above results in case of rational submodules of U, the Utumi ring of quotients of R in place of Martindale ring of quotients Q.

Theorem 4.4.2 Assume that R is a prime ring and U is its Utumi quotient ring. For any rational submodule M of U, the GPIs satisfied by M are the same as the GPIs satisfied by U.

For the proof of the theorem following lemmas are essential.

Lemma 4.4.3 Let N be a rational submodule of U and let $f(x_1, x_2..., x_n)$ be a nontrivial generalized polynomial. Then there exists $a \in N$ such that $f(ax_1, ..., ax_n)a$ is a nontrivial N-generalized polynomial.

Proof Choose a basis B for the C-subspace spanned by the coefficients of a representation of f and write $f = \sum_{i=1}^{s} \alpha_i m_i$, where $\alpha_i \in C\backslash\{0\}$ and where m_i are distinct B-monomials. Note that B is a finite set.

Set $M = (\cap_{i=1}^{s} \alpha_i^{-1} N) \cap N$. M is also a rational submodule of U. By Theorem 4.4.1, there exists $a \in M$ such that $\{ua : u \in B\}$ is a C-linearly independent subset of M.

Consider a B-monomial $m(y_1, ...y_k) = u_0y_1u_1y_2u_2...y_ku_k$, where $\{u_0, ..., u_k\} \subseteq B$ and $\{y_1, ..., y_k\} \subseteq X$. Then $m(ay_1, ...ay_k)a = (u_0a)y_1(u_1a)y_2(u_2a)...y_k(u_ka)$, where $u_0a, u_1a, ..., u_ka \in \{ua : u \in B\}$. Set $B' = \{ua : u \in B\}$. By choice of a, B' is an independent set of M. Hence $m(ay_1, ..., ay_k)a$ is a B'-monomial. Also, if α is one of $\alpha_1, ..., \alpha_s$, then $\alpha m(ay_1, ..., ay_k)a = (\alpha u_0a)y_1(u_1a)y_2(u_2a)...y_k(u_ka)$. By choice of a, $u_0a, u_1a, ..., u_ka \in M \subseteq N$. By the definition of M, $\alpha u_0a \in \alpha M \subseteq N$. Hence $\alpha m(ay_1, ..., ay_k)a$ is an N-generalized polynomial.

Now, consider $f(ax_1, ..., ax_n)a = \sum_{i=1}^{s} \alpha_i m_i(ax_1, ..., ax_n)a$. By the above result, each $m_i(ax_1, ..., ax_n)a$ is a B'-monomial. Hence $f(ax_1, ..., ax_n)a$ is nontrivial. Again, by above result, each $\alpha_i m_i(ax_1, ..., ax_n)a$ is an N-generalized polynomial and hence so is $f(ax_1, ..., ax_n)a$, as desired.

Lemma 4.4.4 Let M be a rational submodule of U. If M satisfies a nontrivial GPI, then R satisfies a nontrivial R-GPI.

Proof Let $f(x_1, ..., x_n) = 0$ be a nontrivial GPI satisfied by M. Set $\rho = M \cap R$. ρ is a rational right ideal of R. By Lemma 4.4.3, there exists $a \in \rho$ such that $f(ax_1, ..., ax_n)a$ is a nontrivial ρ-generalized polynomial. For $r_1, ..., r_n \in R$, $ar_1, ..., ar_n \in \rho R \subseteq \rho \subseteq M$. Hence $f(ar_1, ..., ar_n)a = 0$. So R satisfies the nontrivial ρ-GPI $f(ax_1, ..., ax_n)a = 0$.

Proof of the Theorem 4.4.2 Let M be a rational submodule of U. It is obvious that any GPIs satisfied by U are also satisfied by M. So we show the converse.

If every GPI satisfied by M is trivial, then there is nothing to prove. So we assume that M satisfies a nontrivial GPI. By Lemma 4.4.4, R satisfies a R-GPI. By the main result in [51], the central closure $S(= RC)$ of R contains a minimal idempotent e such that eSe is a finite dimensional division algebra over C. Note that the socle of S is nonzero. By (Proposition 7, [47]) and its proof, the Utumi quotient ring of S is canonically isomorphic to $\mathrm{Hom}(Se, Se)_{eSe}$. Also, under this canonical isomorphism, S is realized via left multiplication as a dense subring of $\mathrm{Hom}(Se, Se)_{eSe}$. From now on, we identify each $s \in S$ with the left multiplication on Se by s. Then the Utumi quotient ring of S is $\mathrm{Hom}(Se, Se)_{eSe}$. Since S is a rational extension of R, the Utumi quotient ring of S coincides with the Utumi quotient ring of R. So we have $U = \mathrm{Hom}(Se, Se)_{eSe}$.

Let $\rho = M \cap R$ and let σ denote the socle of S. Since any GPI is continuous with respect to the finite topology on $\mathrm{Hom}(Se, Se)_{eSe}$ by [30] and since σ is dense in $\mathrm{Hom}(Se, Se)_{eSe}$ with respect to the finite topology, it suffices to show that each GPI satisfied by ρ is also satisfied by σ.

First, suppose that C is finite. Then there exists a rational right ideal ρ' of R such that $\alpha\rho' \subseteq R$ for all $\alpha \in C$. Consider $\rho\rho'$, since $\rho\rho'$ is a rational right ideal of

R, $\rho\rho' C$ is a rational right ideal of S and hence $\rho\rho' C \supseteq \sigma$. But $\rho\rho' C \subseteq \rho R \subseteq \rho$. So $\rho \supseteq \sigma$. Thus any GPI vanishing on ρ also vanishes on σ as is desired.

Now we assume that C is infinite. Let f be a GPI of ρ. We proceed by induction on the height of f to show that f vanishes on U. Pick sufficiently but finitely many distinct $\alpha \in C$. Let ρ' be a rational right ideal of R such that $\alpha\rho' \subseteq \rho$ for all those α we have picked. Replace each indeterminate x in f by αx for these α. Then the resulting GPIs vanish on ρ' So, using the Vandermonde determinant argument, we can solve for the homogeneous parts of f. So each homogeneous part of f vanishes on ρ' It suffices to show that each homogeneous part of f vanishes on U. Note that the height of each homogeneous part of f is less than or equal to that of f. Replacing f by one of its homogeneous parts and ρ by ρ', we may assume from the start that f is homogeneous in each indeterminate it involves.

Assume that the height of f is zero. Then, since f is homogeneous, f must be multilinear By the multilinearty, f vanishes on ρC. But ρC, a rational right ideal of S, must include σ. So f vanishes on σ and hence on U as desired. So we assume that the height of f is larger than 0. As the induction hypothesis, we also assume that the assertion holds for any GPI whose height is less than that of f.

Let x be an indeterminate involved in f. For this moment, we suppress all indeterminates other than x and write $f = f(x)$ for simplicity of notations. Consider $g(x, y) = f(x+y) - f(x) - f(y)$, where y is a new indeterminate not occurring in f. Since g is obviously of less height than f and since g vanishes on ρ, g must vanish U by our induction hypothesis So $f(x+y) = f(x) + f(y)$ for $x, y \in U$. We have thus shown that f is additive on U with respect to each indeterminate it involves.

Now write $f = f(x_1, ..., x_n)$, where $x_1, .., x_n$ are all the indeterminates which f involves. Set $x_i = \sum_j r_j^{(i)}\alpha_j^{(i)}$, where $r_j^{(i)} \in \rho$ and $\alpha_j^{(i)} \in C$. Using the additivity of f on U, we compute

$$f(x_1, ..., x_n) = f(\sum_j r_j^{(1)}\alpha_j^{(1)}, .., \sum_j r_j^{(n)}\alpha_j(n))$$

$$= \sum_{j_1, \ldots, j_n} f(r_{j_1}^{(1)}\alpha_{j_1}^{(1)}, \ldots, r_{j_n}^{(n)}\alpha_{j_n}^{(n)})$$

$$= \sum_{j_1, \ldots, j_n} (\alpha_{j_1}^{(1)})^{h_1} \cdots (\alpha_{j_n}^{(n)})^{h_n} f(r_{j_1}^{(1)}, \ldots, r_{j_n}^{(n)}),$$

where h_i is the x_i-degree of f ($i = 1, \ldots, n$). Since $r^{(i)}_{j_i} \in \rho$, $f(r^{(1)}_{j_1}, \ldots, r^{(n)}_{j_n}) = 0$. Hence $f(x_1, \ldots, x_n) = 0$. But $x_i = \sum_j r^{(i)}_j \alpha^{(i)}_j$ are typical elements of ρC. Since ρC, a rational right ideal of RC, must include σ, f vanishes on σ and hence on U as is desired.

Theorem 4.4.3 Assume that R is a prime ring and U is its Utumi quotient ring. Let M and N be two rational submodules of U. If M satisfies a GPI, then M satisfies a N-GPI.

Proof Let M, N be two given rational submodules of U. In view of Theorem 4.4.2, we may assume that $M = U$. Suppose that $f(x_1, \ldots, x_n) = 0$ is a GPI of U. Then by Lemma 4.4.3, there exists $a \in N$ such that $f(ax_1, \ldots, ax_n)a$ is a nontrivial N-generalized polynomial. Obviously, $f(ax_1, \ldots, ax_n)a = 0$ is also a N-GPI for U.

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